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Analysis of Ricci flow on noncompact manifolds

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To my mother Keya Ren and father Siqin Wu.

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Analysis of Ricci flow on noncompact manifolds

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In this dissertation, we present some analysis of Ricci flow on complete noncompact manifolds.

The first half of the dissertation concerns the formation of Type-II singularity in Ricci flow on $\mathbb{R}^{n+1}$ for $n+1 \geq 3$. For each $\lambda \geq 1$, we construct complete solutions to Ricci flow on $\mathbb{R}^{n+1}$ which encounter global singularities at a finite time T such that the singularities are forming arbitrarily slowly with the curvature blowing up arbitrarily fast at the rate $(T-t)^{-2\lambda}$. Near the origin, blow-ups of such a solution converge uniformly to the Bryant soliton. Near spatial infinity, blow-ups of such a solution converge uniformly to the shrinking cylinder soliton. As an application of this result, we prove that there exist standard solutions of Ricci flow on $\mathbb{R}^{n+1}$ whose blow-ups near the origin converge uniformly to the Bryant soliton.

In the second half of the dissertation, we fully analyze the structure of the Lichnerowicz Laplacian of a Bergman metric g_B on a complex hyperbolic space $\mathbb{CH}^m$ $(m \geq 1)$ and establish the linear stability of the curvature-normalized Ricci flow at such a geometry in complex dimension $m \geq 2$. We then apply the maximal regularity theory for quasilinear parabolic systems to prove a dynamical stability result of Bergman metric on the complete noncompact $\mathbb{CH}^m$ under the curvature-normalized Ricci flow in complex dimension $m \geq 2$. We also prove a similar dynamical stability result on a smooth closed quotient manifold of $(\mathbb{CH}^m, g_B)$ for $m \geq 2$. In order to apply the maximal regularity theory, we define suitably weighted little Hölder spaces on a complete noncompact manifold and establish their interpolation properties.

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Chapter 1

Introduction

Let (M^n, g) be an n-dimensional complete Riemannian manifold. Ricci flow is the evolution of a one-parameter family of metrics $g(t)$ by

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Ric}(g), \quad g(0) = g_0. \tag{1.1}$$

In his seminal paper [43], Hamilton introduced Ricci flow to prove that a compact 3-manifold with positive Ricci curvature is diffeomorphic to a sphere. Since then, Hamilton's Ricci flow program has developed into a powerful analytic tool to understand the geometry and topology of manifolds. Arguably the most celebrated application of Ricci flow is Perelman's proof of Thurston's Geometrization Conjecture [69, 71, 70]. As a corollary, Perelman proved the century-old Poincaré Conjecture.

Conjecture 1.1 (Poincaré, 1904)**.** *Any simply connected closed smooth 3-manifold is diffeomorphic to S^3.*

We refer interested readers to detailed exposition of Perelman's work in the literature, such as [54, 20, 67], and books on the Ricci flow program, including [27, 28, 24, 25, 26].

Another beautiful application of Ricci flow is the proof of the Differential Sphere Theorem by Brendle and Schoen [17].

Theorem 1.1 (Brendle, Schoen)**.** *Let* (M^n, g) *be a compact Riemannian manifold of dimension* $n \geq 4$ *which is strictly* $1/4$*-pinched in the pointwise sense*[1]*. Then* M^n *is diffeomorphic to a spherical space form*[2]*.*

1.1 Ricci flow as a nonlinear PDE

In local coordinates $\{x^i\}_{i=1}^n$, one has

$$-2\,\mathrm{Ric}(g)_{ij} = g^{kl}\left(\frac{\partial^2}{x^i x^j} g_{kl} + \frac{\partial^2}{x^k x^l} g_{ij} - \frac{\partial^2}{x^i x^k} g_{jl} - \frac{\partial^2}{x^j x^l} g_{ik}\right) + \pi(g, g^{-1}, \partial g),$$

where g^{kl} and g^{-1} denote the inverse of g_{kl}, π is a polynomial in g, g^{-1}, and the first derivative of g, and is homogeneous of degree 2 in ∂g. So Ricci flow is a quasilinear PDE.

Because $\mathrm{Ric}(g)$ is invariant under diffeomorphisms, the operator $\mathrm{Ric}(\cdot)$ is *not* elliptic [11, 27], and so Ricci flow is a weakly parabolic PDE. The short-time existence and uniqueness of Ricci flow is known. On a compact manifold, this was first proved by Hamilton [43], and DeTurck gave a simpler proof [34]. On a complete noncompact manifold, Shi established short-time existence [75], and Chen and Zhu proved uniqueness [20].

[1] That is, the sectional curvatures of g satisfy the inequality $0 < \frac{1}{4}K(\pi_1) < K(\pi_2)$ for all points $p \in M^n$ and all 2-planes $\pi_1, \pi_2 \subset T_pM^n$.

[2] A space form is a complete Riemannian manifold of constant sectional curvature.

The long-time existence of Ricci flow, whether M^n is compact or not, is an issue due to the nonlinearity in the equation. For a large family of initial metrics, finite-time singularities in Ricci flow are inevitable. For example, consider the Ricci flow on a compact manifold, then the scalar curvature R evolves under Ricci flow by

$$\begin{aligned} \frac{\partial}{\partial t} R &= \Delta R + 2|\operatorname{Ric}|^2 \\ &\geq \Delta R + \frac{2}{n} R^2, \end{aligned}$$

where Δ is the rough Laplacian[3] $\Delta = g^{ij}\nabla_i\nabla_j$, and $|\operatorname{Ric}|^2 = g^{ij}g^{kl}R_{ik}R_{jl}$. The quadratic nonlinearity R^2 in the above inequality implies that for any initial metric g_0 with strictly positive scalar curvature, the solution $R(t)$ will blow up, and hence Ricci flow will become singular, in finite time.

An essential theme of Ricci flow analysis is that of the singularity formation. A good understanding of the finite-time singularities would enable one to perform Ricci flow with surgery. To describe this process, suppose that one evolves a metric under Ricci flow. One stops the flow once a singularity has formed, and performs a surgery on the evolved manifold by excising any singular region before continuing the flow. Provided that one has sufficient knowledge about the singular regions, one can record the toplogical change in the manifold under surgery, and reconstruct the topology of the original manifold from a limiting flow, together with the singular regions removed.

[3]We use the Einstein summation convention throughout this dissertaion.

Perelman [69, 71] successfully classified finite-time singularities in dimension three and performed Ricci flow with surgery on 3-manifolds, ultimately proving Thurston's Geometrization Conjecutre. Yet, there still remain some major open questions concerning the formation of finite-time singularity in Ricci flow.

1. What curvature blow-up rates and profiles of singular solutions are possible?

2. Which singularity models occur, especially in dimension four?

3. What curvature conditions would allow a classification of the Ricci flow singularities?

In Chapter 2 of this dissertation, we address the first question in the list by exhibiting Ricci flow solutions that encounter the so-called Type-II singularity in finite time on the complete noncompact manifold $\mathbb{R}^{n+1}$ ($n \geq 2$).

Let $g(t)$ be a Ricci flow solution for time $t \geq 0$. Suppose $g(t)$ becomes singular at time $T < \infty$, then the curvature must have become unbounded at T, more precisely ([27, Theorem 6.45]),

$$\lim_{t \nearrow T} \sup_{x \in M^n} |\operatorname{Rm}(x,t)| = \infty.$$

Depending on how fast the Riemann curvature tensor Rm blows up, this finite-time singularity is either *Type-I* if

$$\sup_{M^n \times [0,T)} |\operatorname{Rm}(\cdot,t)| \, (T-t) < \infty,$$

or *Type-II* if

$$\sup_{M^n \times [0,T)} |\mathrm{Rm}(\cdot, t)| \, (T - t) = \infty.$$

The simplest example of a Type-I singularity is that a standard round sphere shrinks to a point under Ricci flow. Hamilton [43] showed that Ricci flow on a compact 3-manifold with positive Ricci curvature develops a Type-I singularity and shrinks to a round point. These Type-I singularities are *global* in the sense that the volume of a manifold shrinks to zero at time T.

Ricci flow singularities can be *local* when a singularity forms on a compact subset of a manifold while the volume of the manifold remains positive at time T. In [46], Hamilton sketched intuitively the formation of *local* singularities under Ricci flow. On a 3-sphere consider an initial geometry resembling that of a dumbbell with a thin neck connecting the two ends, then the manifold will shrink under Ricci flow. In particular, the neck will disappear due to the positive Ricci curvature there, thus forming a so-called *neckpinch*.

There are two possible scenarios of a neckpinch. If the neck disappears before both ends do, then one has a *nondegenerate* neckpinch (see Figure 1.1). If the neck and one end shrink into a cusplike singularity while the other end remains, then one has a *degenerate* neckpinch (see Figure 1.2). The descriptions here are only meant to be intuitive. A rotationally symmetric nondegenerate neckpinch is a Type-I singularity [3]. A degenerate neckpinch is expected, and in many cases confirmed, to be a Type-II singularity [32, 5, 6].

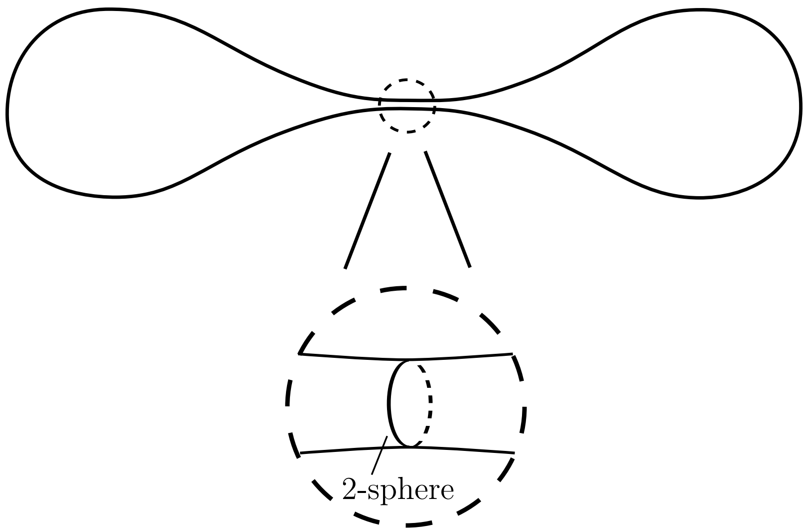

Figure 1.1: Nondegenerate neckpinch in Ricci flow on a 3-sphere and the blow-up of the neck.

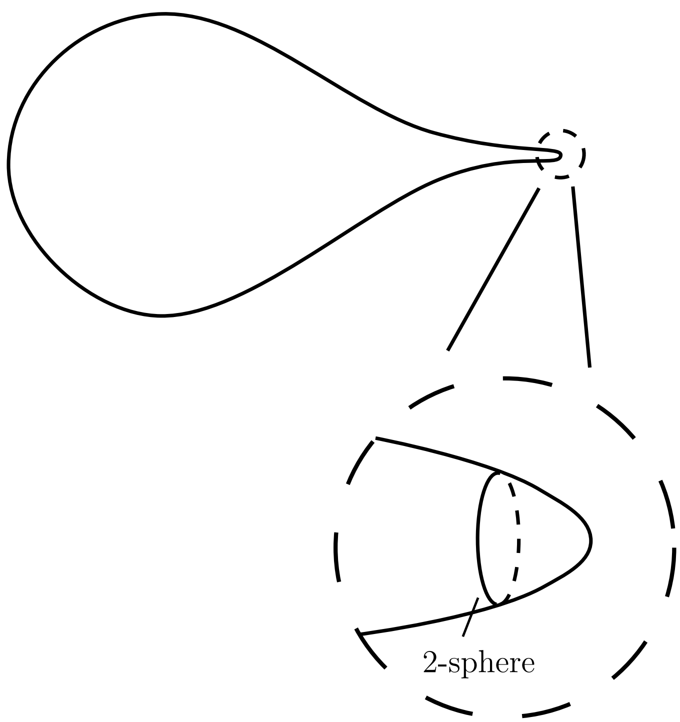

Figure 1.2: Degenerate neckpinch in Ricci flow on a 3-shpere and the blow-up of the tip.

In Chapter 2, we will give rigorous definitions of neckpinch singularities in Ricci flow and review the existing results on Type-I and Type-II singularities modeled by neckpinches. We then construct new solutions that encounter Type-II singularities on $\mathbb{R}^{n+1}$ ($n \geq 2$) that can be regarded as global degenerate neckpinches [83].

1.2 Ricci flow as a dynamical system

It is clear that the fixed points of Ricci flow (1.1) are Ricci-flat metrics. A Ricci-flat metric is an example of an *Einstein* metric, which is defined by the curvature condition

$$\operatorname{Ric}(g) = \lambda g \quad \text{for some } \lambda \in \mathbb{R}.$$

The study of Einstein metrics is a fundamental aspect of geometry; we refer the interested readers to the authoritative reference [11] for details.

One can normalize Ricci flow so that an Einstein metric becomes a fixed point of the normalized flow. When (M^n, g) has finite volume, one can consider the *volume-normalized* Ricci flow:

$$\frac{\partial}{\partial t} g = -2\operatorname{Ric}(g) + \frac{2}{n}\frac{\mathcal{R}(g)}{\int_{M^n} d\mu_g} g, \quad g(0) = g_0, \tag{1.2}$$

where $\mathcal{R}(g) = \int_{M^n} R(g) d\mu_g$ is the total scalar curvature functional. One may also study the *curvature-normalized* Ricci flow, which does not require the manifold to have finite volume:

$$\frac{\partial}{\partial t} g = -2\left(\operatorname{Ric}(g) - \lambda g\right), \quad g(0) = g_0. \tag{1.3}$$

In the rest of this dissertation, we shall refer to equations (1.1), (1.2), and (1.3) collectively as the *Ricci flow*, and we shall specialize only when necessary.

The diffeomorphism invariance of Ricci flow has an intereting consequence. A solution $g(t)$ to equation (1.1) is called a *Ricci soliton* if there exist scalars $\sigma(t)$ and diffeomorphisms ψ_t such that

$$g(t) = \sigma(t)\psi_t^*(g_0), \quad 0 \leq t < T.$$

In fact, $g(t)$ is a Ricci soliton if and only if g_0 satisfies the equation

$$-2\operatorname{Ric}(g_0) = \mathcal{L}_X g_0 + 4\lambda g_0, \tag{1.4}$$

where $\mathcal{L}$ is the Lie derivative, X is some vector field on M, and $\lambda \in \{-1, 0, 1\}$. A soliton is shrinking (or steady, expanding, respectively) if $\lambda = -1$ (or $\lambda = 0, 1$, respectively). If X is the gradient of a function, then the soliton is called a *gradient* soliton. By equation (1.4), Ricci solitons are generalized Einstein metrics. If one regards Ricci flow as a dynamical system on the moduli space of Riemannian metrics modulo diffeomorphism, then Ricci solitons correspond to the generalized fixed points in this context.

Let g_0 be a fixed point of (suitably normalized) Ricci flow. g_0 is said to be *dynamically* stable if the solutions $\tilde{g}(t)$ converge for all choices of initial data $\tilde{g}_0$ that are sufficiently close to g_0 in an appropriate topology. g_0 is said to be *linearly* stable if the spectrum of the elliptic differential operator corresponding to the linearized Ricci flow equation at g_0 is contained in $(-\infty, 0]$.

A corollary of Hamilton's result [43] in dimension three can be viewed as a dynamical stability theorem: if g_0 is any perturbation of a metric of constant positive sectional curvature such that $\mathrm{Ric}(g_0) > 0$, then the volume-normalized Ricci flow starting at g_0 converges exponentially fast in C^∞-topology to a metric of constant positive sectional curvature.

Knowing linear stability, one can prove dynamical stability results using the maximal regularity theory for quasilinear parabolic equations developed by Da Prato and Grisvard [29, 77]. Guenther, Isenberg, and Knopf [41] were the first to employ this approach to prove the dynamical stability for a flat metric. This approach has since then been applied to study the homogeneous Ricci solitons [42], the locally $\mathbb{R}^N$-invariant solutions [55, 80], complex hyperbolic manifolds [82], and extended Ricci flow systems [81]. The linear stability and instability analysis is also fruitful in the study of other geometric flows: for the curve-shortening flow, we have [1] and [35]; for the Willmore flow, there is [78]; and for the cross curvature flow, see [56].

In Chapter 3 of this dissertation, we study the linear stability of Bergman metrics on complex hyperbolic spaces and apply the maximal regularity theory to deduce dynamical stability results [82]. A technicality in the proof is defining suitably weighted spaces on a complete noncompact manifold and establishing the interpolation theory of these weighted spaces.

Chapter 2

Type-II singularities in Ricci flow on $\mathbb{R}^{n+1}$

2.1 Introduction

Recall that a finite-time singularity of a Ricci flow solution $g(t)$, $0 \leq t < T < \infty$, is called Type-I if

$$\sup_{M^n \times [0,T)} |\mathrm{Rm}(\cdot, t)| \, (T-t) < \infty,$$

and it is called Type-II if

$$\sup_{M^n \times [0,T)} |\mathrm{Rm}(\cdot, t)| \, (T-t) = \infty.$$

A shrinking round sphere is an example of a Type-I singularity. The classic result of Hamilton says that Ricci flow on a compact 3-manifold with positive Ricci curvature develops a Type-I singularity and shrinks to a round point [43]. The same is true for Ricci flow on a compact 4-manifold with positive curvature operator, as proved by Hamilton [44]. By the works of Hamilton [45] and Chow [23], Ricci flow on S^2 with an arbitrary initial metric always develops a Type-I singularity and shrinks to a round point. On a compact n-manifold ($n \geq 3$), Böhm and Wilking [12] proved that Ricci flow starting at a metric with 2-positive curvature operator (the sum of the two smallest eigenvalues of Rm is positive) develops a Type-I singularity and shrinks to a round point[1]. Brendle [14] generalized the result of [12] under a much weaker assumption on the curvature operator. All these Type-I singularities are global.

Ricci flow can also encounter finite-time local singularities. For example, Simon [76] showed that there are Ricci flow solutions that encounter finite-time local singularities. The most important example of a local singularity is perhaps the neckpinch described by Hamilton [46], and we have illustrated the intuitive pictures in Section 1.1.

To describe a neckpinch precisely, we recall the blow-up technique in singularity analysis. We say that a sequence $\{(x_i, t_i)\}_{i=0}^{\infty}$ of points and times in a Ricci flow is a *blow-up sequence* at time T if $t_i \nearrow T$ and $|\operatorname{Rm}(x_i, t_i)| \nearrow \infty$ as $i \nearrow \infty$. A blow-up sequence has a *pointed singularity model* if the sequence of parabolically dilated metrics

$$g_i(x,t) := |\operatorname{Rm}(x_i,t_i)|\, g\left(x, t_i + |\operatorname{Rm}(x_i,t_i)|^{-1} t\right)$$

has a complete smooth limiting metric. A Ricci flow solution is said to develop a neckpinch singularity at time $T < \infty$ if there is some blow-up sequence at T whose pointed singularity model exists and is given by the self-similarly shrinking Ricci soliton on the cylinder $\mathbb{R} \times S^n$.

A neckpinch singularity is *nondegenerate* if every pointed singularity model of any blow-up sequence at T is a shrinking cylinder soliton. A nondegenerate neckpinch is a Type-I singularity. The first rigorous examples of

[1]We note that the $n = 4$ case in their result was known earlier [21].

finite-time neckpinch singularities in Ricci flow on a compact manifold were produced by Angenent and Knopf [3]. They exhibited a class of rotationally symmetric metrics on S^{n+1} $(n \geq 2)$ which develop Type-I neckpinch singularities under Ricci flow. In a subsequent paper [7], the same authors proved the precise asymptotics for such neckpinch singularities.

A neckpinch singularity is *degenerate* if there is at least one blow-up sequence at T with a pointed singularity model that is not a shrinking cylinder soliton. A degenerate neckpinch is expected to be a Type-II singularity. Daskalopoulos and Hamilton [32] showed that on $\mathbb{R}^2$ there exist complete noncompact Ricci flow solutions that form Type-II singularities at the rate $(T-t)^{-2}$. Their proof is particular to dimension two, in which case the Ricci flow is conformal. Indeed, by the Uniformization Theorem, every metric g on $\mathbb{R}^2$ is conformal to the Euclidean flat metric δ_E by $g = u\delta_E$, and hence Ricci flow is equivalent to the evolution of the conformal factor u by the logarithmic fast diffusion equation $u_t = \Delta \log u$. Assuming rotational symmetry and additional constraints, Daskalopoulos and del Pino [31] gave a precise description of the extinction profile of this maximal solution in $\mathbb{R}^2$: up to proper scaling, it must be the cigar soliton[2] in an inner region, and a logarithmic cusp in an outer region. Daskalopoulos and Šešum [33] proved the same result without assuming rotational symmetry. An extension of the results of [31, 33] was obtained by Hui [49]. The formal asymptotics of the extinction profile were

[2]The cigar soliton is the complete Riemannian surface $(\mathbb{R}^2, g_\Sigma)$, where $g_\Sigma = \frac{dx^2+dy^2}{1+x^2+y^2}$.

derived by King [53].

In dimension three or higher, if one is willing to assume rotational symmetry of the metrics, then the Ricci flow is reduced to a parabolic equation for a scalar function. Gu and Zhu [40] proved the existence of Type-II singularities on S^{n+1}, although their work shed little light on the geometric details of such solutions. Garfinkle and Isenberg [36, 37] have conducted numerical investigations on the formation of Type-II singularities modeled by degenerate neckpinches on S^3. In their recent works, Angenent, Isenberg, and Knopf [5, 6] demonstrated the existence of rotationally symmetric Ricci flow solutions on S^{n+1} that develop finite-time Type-II degenerate neckpinches. Their solutions become singular at the rate $(T-t)^{-2+2/k}$ for $k \in \mathbb{N}$ and $k \geq 3$. Moreover, they were able to describe the asymptotic profiles of these solutions.

We note that Ricci flow on $\mathbb{R}^{n+1}$ $(n \geq 1)$ can encounter finite-time singularity. For example, take a metric on S^{n+1} as constructed in [3] and conformally open up the north pole of the sphere. This produces an initial geometry on $\mathbb{R}^{n+1}$, which one expects to develop finite-time Type-I neckpinch singularity under Ricci flow. Similarly, one expects that there are Ricci flow solutions that form finite-time Type-II singularities on $\mathbb{R}^{n+1}$. Indeed, this happens on $\mathbb{R}^2$ [32].

In this chapter, we construct rotationally symmetric Ricci flow solutions that encounter finite-time Type-II singularities on $\mathbb{R}^{n+1}$ $(n \geq 2)$ that can be regarded as global degenerate neckpinches. At time T, these solutions are not

κ-noncollapsed at all scales[3], and hence they cannot be blow-ups of Ricci flow singularities on a compact manifold [69].

We now state the main result in this chapter.

Theorem 2.1. *Let $\lambda \geq 1$. In each dimension $n+1 \geq 3$, there exists an open set of complete rotationally symmetric metrics $\mathcal{G}_{n+1}$ on $\mathbb{R}^{n+1}$ such that the Ricci flow starting at $g_0 \in \mathcal{G}_{n+1}$ has a unique solution $g(t)$ for $t \in [0,T)$, $T < \infty$. The solution $g(t)$ develops a finite-time global singularity at time T with the following properties.*

1. *The singularity is Type-II with*

$$\sup_{x\in\mathbb{R}^{n+1}} |\operatorname{Rm}(x,t)| = \frac{C}{(T-t)^{2\lambda}}$$

   attained at the origin, where C is a constant depending on n.

2. *If one rescales a solution so that the distance from the origin dilates at the rate $(T-t)^{-\lambda}$, then the metric converges uniformly on intervals of order $(T-t)^{\lambda}$ to the Bryant soliton.*

3. *If one rescales a solution at the parabolic rate $(T-t)^{-1/2}$, then the metric converges uniformly to the shrinking cylinder soliton near spatial infinity.*

Furthermore, the solutions exhibit the asymptotic behavior of the formal solution described in Section 2.3.

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[3]On a complete Riemannian manifold (M^n, g), given $\rho \in (0,\infty]$ and $\kappa > 0$, we say that the metric g is κ-noncollapsed below the scale ρ if for any metric ball $B(x,r)$ with $r < \rho$ satisfying $|\operatorname{Rm}(y)| \leq r^{-2}$ for all $y \in B(x,r)$, we have $\operatorname{Vol}(B(x,r)) \geq \kappa r^n$. If g is κ-noncollapsed below the scale ∞, we say that g is κ-noncollapsed at all scales.

Remark 2.1. *The singular time T is determined only by the initial radius of the asymptotic cylinder at spatial infinity. In terms of the rescaled time τ_0 (cf. Proposition 2.6), $T = e^{-\tau_0}$.*

Theorem 2.1 is inspired by [5, 6]. To prove it, we begin by constructing a formal solution to Ricci flow on $\mathbb{R}^{n+1}$ with curvature blow-up rate of $(T-t)^{-2\lambda}$ near the origin for each $\lambda \geq 1$, and of $(T-t)^{-1}$ near spatial infinity. Using the formal solution, we construct upper and lower barriers to the Ricci flow PDE and prove a comparison principle. Before the first singular time T, the curvatures are bounded and so the Ricci flow solution exists and is unique [75, 20]. For all initial data between the barriers, we obtain unique complete solutions to the Ricci flow whose asymptotic properties are the same as those of the formal solution.

Our result is interesting in several aspects. First of all, solutions in Theorem 2.1 provide the only examples of Type-II singularities on $\mathbb{R}^{n+1}$, $n \geq 2$. They show that Type-II singularities on $\mathbb{R}^{n+1}$ can occur arbitrarily slowly[4] with curvatures blowing up at arbitrarily fast rate of $(T-t)^{-2\lambda}$, $\lambda \geq 1$. The $\lambda = 1$ case in Theorem 2.1 can be viewed as a higher-dimensional version of the result of Daskalopolous and Hamilton [32] for rotationally symmetric solutions. The asymptotics in Theorem 2.1 can be compared to those in [31, 33]. Secondly, solutions in [6] become singular at the set of discrete rates $(T-t)^{-2+2/k}$, where

[4]The singularity is slowly forming in the following sense: for every $C > 0$, there exists a time $t_C < T$ such that $T - t_C > C/\sup_{x\in\mathbb{R}^n} |\mathrm{Rm}(x, t_C)|$, i.e. there is more time remaining until extinction than the maximum curvature predicts.

$k \in \mathbb{N}$ and $k \geq 3$. In contrast, the curvature blow-up rates of the Ricci flow solutions in Theorem 2.1 form a continuum since $\lambda \in [1, \infty)$. In particular, the $\lambda = 1$ case can be thought of as the limiting case of [6, Main Theorem] as $k \nearrow \infty$. Thirdly, the analysis in [36, 37, 40] suggest that the formation of Type-II singularity on a compact manifold is an unstable property. As a result, the proof in [6] uses the somewhat indirect Ważewski retraction method. In comparison, we use a comparison principle to give a direct proof of Theorem 2.1. So one may regard the formation of Type-II singularity on a noncompact manifold to be a stable property.

Our solutions are modeled by the Bryant soliton near the origin. This is reasonable because blow-ups of Ricci flow singularities are expected, and in many cases proved, to be Ricci solitons. On $\mathbb{R}^{n+1}$ ($n \geq 2$), the Bryant soliton is a rotationally symmetric gradient steady Ricci soliton with positive curvature operator [18, 28]. In dimension three, Bryant [18] showed that there are no other rotationally symmetric steady Ricci solitons. Other non-rotationally symmetric solitons exist in higher dimensions by the work of Ivey [51]. Perelman [69] asked if any three-dimensional steady Ricci soliton is necessarily rotationally symmetric. Brendle [16] answered this question affirmatively by proving that on $\mathbb{R}^3$, any steady gradient Ricci soliton which is κ-noncollapsed and non-flat must be rotationally symmetric. Brendle [15] also proved a higher dimensional version of this theorem. There are other uniqueness results for Bryant solitons under the additional assumption such as local conformal flatness [19], or suitable asymptotics near spatial infinity [13], or half conformal flatness [22].

In [71], Perelman described a special family of Ricci flow solutions on $\mathbb{R}^3$, the so-called *standard solutions*, which are complete rotationally symmetric metrics that are asymptotic to a round cylinder at spatial infinity. The standard solutions are used to construct long-time solutions and to study Ricci flow with surgery [71, 54]. We will see that a subset of Ricci flow solutions in Theorem 2.1 are in fact standard solutions in the sense defined in [63]. More precisely, we have the following result. We refer the reader to Section 2.7 for the definition of a standard solution.

Theorem 2.2. *Let $n+1 \geq 3$. Let $\mathcal{G}_{n+1}$ be given as in Theorem 2.1. There exists an open set $\mathcal{G}^*_{n+1} \subset \mathcal{G}_{n+1}$ such that the Ricci flow starting at $g_0 \in \mathcal{G}^*_{n+1}$ has a unique standard solution $g(t)$ on $\mathbb{R}^{n+1}$ for $t \in [0, T)$, $T < \infty$. Moreover, the solution $g(t)$ satisfies all the properties described in Theorem 2.1.*

Consequently, we have the following result.

Corollary 2.2. *In dimension three or higher, there exist standard solutions to Ricci flow whose blow-ups near the origin converge uniformly (cf. part (2) of Theorem 2.1) to the Bryant soliton.*

Let $n+1 \geq 3$. Let $\mathcal{G}_{n+1}$ be given as in Theorem 2.1. There exists an open set $\mathcal{G}^*_{n+1} \subset \mathcal{G}_{n+1}$ such that the Ricci flow starting at $g_0 \in \mathcal{G}^*_{n+1}$ has a unique standard solution $g(t)$ on $\mathbb{R}^{n+1}$ for $t \in [0, T)$, $T < \infty$. Moreover, the solution $g(t)$ satisfies all the properties described in Theorem 2.1. Bennett

Chow and Gang Tian have conjectured that in dimension three or higher, blow-ups near the origion of a standard solution to Ricci flow converge to the Bryant soliton in some suitable topology [62]. We see that Corollary 2.2 gives evidence in favor of the Chow-Tian conjecture.

Chapter 2 is organized as follows. In Section 2.2, we describe the basic set-up and the coordinates which we will use. In Section 2.3, we construct the formal solution with matched asymptotics. We construct sub- and supersolutions to the Ricci flow PDE in Section 2.4, and use these functions to construct upper and lower barriers in Section 2.5. In Section 2.6, we prove a comparison principle for the Ricci flow PDE and use it to prove Theorem 2.1. In Section 2.7, we relate our solutions to the standard solutions and prove Theorem 2.2.

2.2 Preliminaries

Let g_{sph} be the metric of constant sectional curvature one on S^n. We puncture $\mathbb{R}^{n+1}$ at the origin and identify the remaining manifold with $(0,\infty)\times S^n$. For $x \in (0,\infty)$, we define a warped product metric

$$g = \varphi^2(x)dx^2 + \psi^2(x)g_{\text{sph}}.$$

The distance s to the origin is

$$s(t,x) := \int_0^x \varphi(t,y)dy.$$

In the s-coordinate, the metric becomes

$$g = ds^2 + \psi^2\left(s,t\right)g_{\text{sph}}. \tag{2.1}$$

Extending the metric g to a complete smooth rotationally symmetric metric, which we still denote by g, on $\mathbb{R}^{n+1}$, the function ψ necessarily satisfies the boundary conditions at $x = 0$ [27, Lemma 2.10]:

$$\lim_{x \searrow 0} \psi = 0 \quad \text{and} \quad \lim_{x \searrow 0} \psi_s = 1.$$

We let $\partial_t|_x$ and $\partial_t|_s$ denote taking time derivatives while keeping x and s fixed, respectively. Then

$$[\partial_t|_x, \partial_s] = -n\frac{\psi_{ss}}{\psi}\partial_s.$$

In the s-coordinate, the Ricci flow is reduced to the scalar equation ([3, Equation (10)])

$$\partial_t|_x \psi = \psi_{ss} - (n-1)\frac{1-\psi_s^2}{\psi^2}. \tag{2.2}$$

The function φ, which is suppressed in the s-coordinate, evolves by the equation ([3, Equation (11)])

$$\partial_t|_x(\log \varphi) = n\frac{\psi_{ss}}{\psi}.$$

Let K be the sectional curvature of a two-plane orthogonal to the sphere $\{x\} \times S^n$, and let L be the sectional curvature of a tangential two-plane. Then

$$K = -\frac{\psi_{ss}}{\psi}, \quad L = \frac{1-\psi_s^2}{\psi^2}.$$

In particular, $|\operatorname{Rm}|^2 = 2nK^2 + n(n-1)L^2$.

Since $\lim_{x\searrow 0}\psi_s = 1$ and the metric is smooth, $\psi_s > 0$ in a neighborhood of the origin. So we can use ψ as a new coordinate there, writing

$$g = z(\psi,t)^{-1}d\psi^2 + \psi^2 g_{\mathrm{sph}}, \tag{2.3}$$

where $z\,(\psi,t) := \psi_s^2$. Under the Ricci flow, cf. [6, Section 2.2], the metric (2.3) evolves by

$$\partial_t|_\psi z = \mathcal{E}_\psi[z], \tag{2.4}$$

where $\mathcal{E}_\psi$ is the purely local quasilinear operator

$$\mathcal{E}_\psi[z] := zz_{\psi\psi} - \frac{1}{2}z_\psi^2 + (n-1-z)\frac{z_\psi}{\psi} + 2(n-1)\frac{(1-z)z}{\psi^2}.$$

We can split $\mathcal{E}_\psi$ into a linear and a quadratic term:

$$\mathcal{E}_\psi[z] = \mathcal{L}_\psi[z] + \mathcal{Q}_\psi[z],$$

where

$$\mathcal{L}_\psi[z] := (n-1)\left(\frac{z_\psi}{\psi} + 2\frac{z}{\psi^2}\right), \tag{2.5}$$

$$\mathcal{Q}_\psi[z] := zz_{\psi\psi} - \frac{1}{2}z_\psi^2 - \frac{zz_\psi}{\psi} - 2(n-1)\frac{z^2}{\psi^2}. \tag{2.6}$$

The quadratic part defines a symmetric bilinear operator

$$\begin{aligned}\hat{\mathcal{Q}}_\psi[z_1,z_2] := {}& \frac{1}{2}\left[z_1(z_2)_{\psi\psi} + z_2(z_1)_{\psi\psi} - (z_1)_\psi(z_2)_\psi\right] \\ & - \frac{z_1(z_2)_\psi + z_2(z_1)_\psi}{2\psi} - 2(n-1)\frac{z_1z_2}{\psi^2}.\end{aligned} \tag{2.7}$$

In particular, $\mathcal{Q}_\psi[z] = \hat{\mathcal{Q}}_\psi[z,z]$.

Throughout this chapter, we use C_k ($k \in \mathbb{N}$) to denote a constant that may change from line to line. The expression "$f \lesssim g$" means $f \le C_k g$ for some constant C_k.

2.3 The formal solution

We first briefly review the formal solution in [5, 6]. Introducing the coordinates consistent with a parabolic cylindrical blow-up (see for exmaple, [6, Section 3.1]):

$$u := \frac{\psi}{\sqrt{2(n-1)(T-t)}}, \quad \sigma := \frac{s}{\sqrt{T-t}}, \quad \tau := -\log(T-t), \tag{2.8}$$

then in these coordinates, equation (2.2) becomes

$$\partial_\tau|_\sigma u = u_{\sigma\sigma} - \left(\frac{\sigma}{2} + nI[u]\right) u_\sigma + \frac{u - u^{-1}}{2} + (n-1)\frac{u_\sigma^2}{u}, \tag{2.9}$$

where

$$I[u](\sigma,\tau) := \int_0^\sigma \frac{u_{\hat{\sigma}\hat{\sigma}}(\hat{\sigma},\tau)}{u(\hat{\sigma},\tau)} d\hat{\sigma}.$$

For bounded σ, the solution to equation (2.9) is approximated by

$$U \approx 1 + \sum_{m=0}^{\infty} a_m e^{(1-m/2)\tau} h_m(\sigma),$$

where h_m is the m-th Hermite polynomial. In [5, 6], the authors assume a nondegenerate neckpinch occurs at the equator of S^{n+1} in such a way that the Ricci flow solution *does not* approach a cylinder too quickly. So the term with $m = k$ is dominant for some specified $k \geq 3$. They then construct a formal solution with matched asymptotics in four connected regions: the outer, parabolic, intermediate, and tip regions. Their construction starts in the parabolic region which models a nondegenerate neckpinch near the equator of S^{n+1}, and ends in the tip region which models a degenerate neckpinch at one of the poles of S^{n+1}.

In this chapter, we are interested in solutions that approach a cylinder "quickly" in contrast to [6, Section 3.1]. This leads us to the following construction. We first build a model for a degenerate neckpinch near the origin of $\mathbb{R}^{n+1}$, and we then work our way out to the rest of the manifold. We will see that our formal solution is defined in two connected regions: the *interior* and the *exterior* regions. It turns out, cf. the proof of Lemma 2.13, these are enough to define complete metrics on $\mathbb{R}^{n+1}$. One may compare to the compact case and think that in the noncompact case the parabolic and the outer regions are pushed to spatial infinity.

2.3.1 Approximate solution in the interior region

In the interior region, which is to be specified in (2.23), we expect u to be small and introduce the variable

$$r := e^{\gamma\tau}u,$$

where $\gamma > 0$ is a constant to be specified.

In the u-coordinate, by the change of variable formulae:

$$\partial_t|_\psi z = \{\partial_\tau|_u z + z_u\left(\partial_\tau|_\psi u\right)\}\frac{d\tau}{dt} = \left(\partial_\tau|_u z + \frac{1}{2}\psi z_\psi\right)e^\tau,$$
$$\mathcal{E}_\psi[z] = \frac{1}{2(n-1)}e^\tau\mathcal{E}_u[z],$$

equation (2.4) becomes

$$\partial_\tau|_u z = \frac{1}{2(n-1)}\mathcal{E}_u[z] - \frac{1}{2}uz_u, \tag{2.10}$$

where we have used $\psi z_\psi = u z_u$.

In the r-coordinate, since

$$\partial_\tau|_r z = \partial_\tau|_u z + z_u\left(\partial_\tau|_r u\right) = \partial_\tau|_u z - \gamma u z_u, \quad u z_u = r z_r,$$
$$\mathcal{E}_u[z] = e^{2\gamma\tau}\mathcal{E}_r[z],$$

equation (2.10) becomes

$$\mathcal{T}_r[z] = 0,$$

where

$$\mathcal{T}_r[z] := e^{-2\gamma\tau}\left\{\partial_\tau|_r z + \left(\frac{1}{2} + \gamma\right) r z_r\right\} - \frac{1}{2(n-1)}\mathcal{E}_r[z]. \tag{2.11}$$

For sufficiently large τ, the term involving $e^{-2\gamma\tau}$ becomes neglible and the equation $\mathcal{T}_r[z] = 0$ is approximated by

$$\mathcal{E}_r[z] = 0,$$

whose solutions are the Bryant soliton profile functions

$$z(r) = \mathfrak{B}(ar),$$

where $a > 0$ is an arbitrary constant. Each member of the one-parameter family of complete smooth metrics given by

$$g = \mathfrak{B}^{-1}(ar)dr^2 + r^2 g_{\mathrm{sph}}$$

is a scaled version of the Bryant soliton.

The function $\mathfrak{B}(r)$ is smooth and strictly monotone decreasing for all $r > 0$. Near $r = 0$, $\mathfrak{B}(r)$ has the asymptotic expansion

$$\mathfrak{B}(r) = 1 - b_2 r^2 + b_3 r^4 + b_3 r^6 + \cdots \quad \text{as } r \searrow 0, \tag{2.12}$$

where b_k's are constants. In particular, $b_2 > 0$. Near $r = \infty$, $\mathfrak{B}(r)$ has the asymptotic expansion

$$\mathfrak{B}(r) = r^{-2} + c_2 r^{-4} + c_3 r^{-6} + \cdots \quad \text{as } r \nearrow \infty, \tag{2.13}$$

where c_k's are constants. In this chapter, we normalize $\mathfrak{B}(r)$ by setting $c_2 = 1$. For more information on $\mathfrak{B}(r)$, we refer the reader to [5, Appendix B].

We refine the approximate solution by considering an expansion of the form

$$z = \mathfrak{B}(ar) + e^{-\tilde{\gamma}\tau}\beta_1(r) + e^{-2\tilde{\gamma}\tau}\beta_2(r) + \cdots, \tag{2.14}$$

where $\tilde{\gamma} > 0$, then

$$z \sim a^{-2} r^{-2} \quad \text{as } \tau \nearrow \infty,\ r \nearrow \infty,$$

which, in terms of the u-coordinate, is

$$z \sim a^{-2} e^{-2\gamma\tau} u^{-2} \quad \text{as } \tau \nearrow \infty,\ u \text{ small}.$$

In Lemma 2.4, we will use $z = \mathfrak{B}(ar) + e^{-\tilde{\gamma}\tau}\beta_1(r)$ with $\tilde{\gamma} = \lambda$ to construct sub- and supersolutions in the interior region.

2.3.2 Approximate solution in the exterior region

We expect the exterior region, which is to be specified in (2.29), to be a time-dependent subset of the neighborhood of the origin where $1 > z > 0$ and $0 < u < 1$. In this region, z evolves by equation (2.10), i.e.,

$$\partial_\tau|_u z = \frac{1}{2(n-1)}\mathcal{E}_u[z] - \frac{1}{2}uz_u.$$

To construct a formal solution to this equation, we try the series

$$z = e^{-\lambda\tau}Z_1(u) + e^{-2\lambda\tau}Z_2(u) + \cdots = \sum_{m\geq 1} e^{-m\lambda\tau}Z_m(u), \tag{2.15}$$

where $\lambda > 0$ is a constant to be chosen. We substitute this expansion into the equation above and split $\mathcal{E}_u[z]$ into linear and quadratic parts given in (2.5) and (2.6) respectively. By comparing the coefficient of $e^{-m\lambda\tau}$ in the resulting equation, we find Z_m must satisfy the ODE

$$\frac{1}{2}(u^{-1} - u)\frac{dZ_m}{du} + \left(u^{-2} + m\lambda\right) Z_m = -\frac{1}{2(n-1)}\sum_{i=1}^{m-1}\hat{\mathcal{Q}}_u[Z_i, Z_{m-i}]. \tag{2.16}$$

When $m = 1$, equation (2.16) is a linear homogeneous equation

$$\frac{1}{2}(u^{-1} - u)\frac{dZ_1}{du} + \left(u^{-2} + \lambda\right) Z_1 = 0, \tag{2.17}$$

whose solutions are

$$Z_1(u) = bu^{-2}(1 - u^2)^{1+\lambda}, \tag{2.18}$$

where b is an arbitrary constant that will be determined by matching considerations.

When $m = 2$, equation (2.16) becomes

$$\frac{1}{2}(u^{-1} - u)\frac{dZ_2}{du} + \left(u^{-2} + 2\lambda\right) Z_2 = \mathcal{Q}_u[Z_1], \tag{2.19}$$

where

$$\begin{aligned}\mathcal{Q}_u[Z_1] = 2b^2 u^{-6}(1 - u^2)^{2\lambda}\big\{4 - n(1 - u^2)^2 \\ + 2u^2(\lambda - 3) + u^4(\lambda - 1)^2\big\}.\end{aligned} \tag{2.20}$$

So the solutions of equation (2.19) are

$$Z_2(u) = u^{-2}(1 - u^2)^{1+2\lambda} f(u),$$

where

$$\begin{aligned}f(u) = C_1 - 2b^2 \left(\frac{4 - n}{u^2} - \frac{\lambda^2 - 1}{1 - u^2}\right) \\ - 4(1 + \lambda)b^2 \left(\log(1 - u^2) - 2\log u\right)\end{aligned}$$

for an arbitrary constant C_1.

By direction computation, we have the following.

Lemma 2.3. *If $\lambda \geq 1$, then*

$$\lim_{u \nearrow 1} \Big| Z_2(u)/Z_1(u) \Big| = 0.$$

So for any $\lambda \geq 1$,

$$z(u, \tau) \approx e^{-\lambda\tau} b u^{-2}(1 - u^2)^{1+\lambda}$$

is a valid approximation for $u \nearrow 1$ and τ sufficiently large. Going in the other direction, as $u \searrow 0$,

$$Z_1(u) \approx bu^{-2}, \quad Z_2(u) = O(u^{-4}),$$

so then

$$z \approx e^{-\lambda\tau} bu^{-2} + O\left(e^{-2\lambda\tau} u^{-4}\right).$$

This approximation is valid as long as

$$|e^{-\lambda\tau} bu^{-2}| \gg |e^{-2\lambda\tau} u^{-4}|,$$

which is when

$$u \gg e^{-\lambda\tau/2},$$

or equivalently, in the r-coordinate,

$$r \gg e^{(\gamma-\lambda/2)\tau}.$$

From now on, given $\lambda \geq 1$, we choose $\gamma = \lambda/2$. We do not need to consider the terms Z_m for $m \geq 2$ by the barrier argument in Section 2.5.

2.3.3 Matching condition

We now match the formal solutions in the interior and the exterior regions when τ is sufficiently large. At $r = A \gg 1$, the formal solution in the interior region is approximately

$$
\begin{aligned}
z(A) &\approx \mathfrak{B}(aA) + e^{-\lambda\tau/2}\beta(A) \\
&\approx \mathfrak{B}(aA) \\
&\approx (aA)^{-2}.
\end{aligned}
$$

At $u = e^{-\lambda\tau/2}A$, the formal solution in the exterior region is approximately

$$
\begin{aligned}
z\left(e^{-\lambda\tau/2}A\right) &\approx e^{-\lambda\tau} Z_1\left(e^{-\lambda\tau/2}A\right) \\
&\approx bA^{-2}(1 - e^{-\lambda\tau}A^2)^2 \\
&\approx bA^{-2}.
\end{aligned}
$$

Thus, matching the two expressions implies that for a given constant $a > 0$, we ought to have

$$
b \approx a^{-2}.
$$

This relation is made more precise in Lemma 2.9.

2.3.4 Features of the formal solution

Our formal solution z_{form} is valid for all dimensions $n+1 \geq 3$, and it is defined in the interior and the exterior regions as follows: for each $\lambda \geq 1$,

$$
z_{\text{form}} = \left\{ \begin{array}{cc} \mathfrak{B}(ar),\ a > 0, & \text{in the interior region,} \\ e^{-\lambda\tau} bu^{-2}(1-u^2)^{1+\lambda},\ b \approx a^{-2}, & \text{in the exterior region.} \end{array} \right. \tag{2.21}
$$

Cf. the proof of Lemma 2.13, the metric corresponding to the formal solution is complete on $\mathbb{R}^{n+1}$, and one approaches spatial infinity as $u \nearrow 1$.

Also, as $u \nearrow 1$, $z(u) \searrow 0$, i.e., $\psi_s \searrow 0$, so the metric (2.1) is approaching that of a round cylinder near spatial infinity. As $u \searrow 0$ and $\tau \nearrow \infty$, $z(u) \nearrow 1$ and the formal solution z is asymptotic to a Bryant soliton profile function.

The norm of the curvature tensor achieves its maximum value at the origin O [6], where we have

$$|\operatorname{Rm}(O,t)| = \frac{C}{(T-t)^{2\lambda}}$$

for some constant C depending on n. Thus, the curvature of a Ricci flow solution that asymptotically approaches this formal solution necessarily blows up at the same rate.

2.4 Sub- and supersolutions

A metric of the form (2.3) evolving under the Ricci flow is determined by a profile function z which, in the u-coordinate, satisfies the quasilinear parabolic equation (2.10). In this section, we construct sub- and supersolutions to this equation in the interior and the exterior regions, respectively.

2.4.1 In the interior region

In the r-coordinate, where $r = e^{\lambda\tau/2}u$, z satisfies the equation $\mathcal{T}_r[z] = 0$, where the operator $\mathcal{T}_r$ is defined in (2.11). We call z a subsolution (supersolution) of $\mathcal{T}_r[z] = 0$ if $\mathcal{T}_r[z] \leq 0$ (≥ 0).

Lemma 2.4. *Let $\lambda \geq 1$. For any $A_1 > 0$, there exist a bounded function $\beta : (0,\infty) \to \mathbb{R}$, a sufficiently small $B_1 > 0$, and a sufficiently large $\tau_1 < \infty$,* *all depending only on A_1 such that the functions*

$$z^{\pm}_{int} := \mathfrak{B}(A_1 r) \pm e^{-\lambda\tau}\beta(r) \tag{2.22}$$

are sub- (z^{-}_{int}) and super- (z^{+}_{int}) solutions of $\mathcal{T}_r[z] = 0$ in the interior region

$$\Omega_{int} := \{0 \le r \le B_1 e^{\lambda\tau/2}\} \tag{2.23}$$

for all $\tau \ge \tau_1$.

Proof. Let $\mathbf{B}(r) := \mathfrak{B}(A_1 r)$. For $z(r) = \mathbf{B}(r) + e^{-\lambda\tau}\beta(r)$ to be a supersolution, it suffices to show $\mathcal{T}_r[z] \ge 0$. Since $\mathbf{B}(r)$ solves $\mathcal{E}_r[z] = 0$,

$$\begin{aligned}\mathcal{T}_r\left[z^{+}_{\text{int}}\right] =& e^{-\lambda\tau}\left\{-\frac{\mathcal{L}_r[\beta] + 2\hat{\mathcal{Q}}_r[\mathbf{B},\beta]}{2(n-1)} + \frac{\lambda+1}{2} r\mathbf{B}'\right\} \\ &+ e^{-2\lambda\tau}\left\{-\lambda\beta + \frac{\lambda+1}{2} r\beta' - \frac{\mathcal{Q}_r[\beta]}{2(n-1)}\right\}.\end{aligned}$$

Set $\hat{A} := 1 + \frac{\lambda+1}{2}$, and let β solve the linear inhomogeneous ODE

$$\mathcal{L}_r[\beta] + 2\hat{\mathcal{Q}}_r[\mathbf{B},\beta] = 2(n-1)\hat{A} r\mathbf{B}'. \tag{2.24}$$

Using the definitions of $\mathcal{L}_r$ and $\hat{\mathcal{Q}}_r$ in (2.5) and (2.6) respectively, equation (2.24) becomes

$$\begin{aligned}\mathbf{B}\beta'' + &\left\{\frac{n-1}{r} - \mathbf{B}' - \frac{\mathbf{B}}{r}\right\}\beta' \\ &+ \left\{\mathbf{B}'' - \frac{\mathbf{B}'}{r} + 2(n-1)\frac{1-2\mathbf{B}}{r^2}\right\}\beta = 2(n-1)\hat{A} r\mathbf{B}'.\end{aligned} \tag{2.25}$$

Recall the asymptotic expansions of $\mathbf{B}(r)$ near $r = 0$ and $r = \infty$ given by (2.12) and (2.13), respectively. Then near $r = 0$, equation (2.25) is approximated by

$$\beta'' + \frac{n-2}{r}\beta' - \frac{2(n-1)}{r^2}\beta = -C_1 r^2 \quad (C_1 > 0),$$

whose solution is

$$\beta_0 = C_2 r^{1-n} + C_3 r^2 - C_4 r^4,$$

where C_2, C_3 are arbitrary constants and C_4 is a constant depending on C_1. Discarding the unbounded solution and choosing $C_3 = 1$, then there exists a solution β_p of equation (2.24) with

$$\beta_p(r) = r^2 + o(r^2) \quad \text{as } r \searrow 0.$$

Near $r = \infty$, the ODE (2.25) is a perturbation of the equation

$$\frac{1}{(A_1 r)^2}\beta'' + \frac{n-1}{r}\beta' + \frac{2(n-1)}{r^2}\beta = -\frac{4(n-1)\hat{A}}{(A_1 r)^2},$$

whose general solution is

$$\beta_\infty(r) = C_5 r e^{-\alpha r^2} + C_6 r \int_1^r \rho^{-2} e^{-\alpha(r^2-\rho^2)} d\rho - \frac{2\hat{A}}{A_1^2},$$

with $\alpha := \frac{n-1}{2}A_1^2$ and arbitrary constants C_5, C_6. The second term in this expression is $O\left(r^{-2}\right)$. So every solution of equation (2.24), in particular $\beta_p(r)$ given above, has the following asymptotic expansions:

$$\beta(r) = \begin{cases} r^2 + o(r^2) & \text{as } r \searrow 0, \\ -2\hat{A}/A_1^2 + o(1) & \text{as } r \nearrow \infty. \end{cases} \tag{2.26}$$

Also, the asymptotic expansions

$$-r\mathbf{B}'(r) = \begin{cases} C_7 r^2 + o(r^2) & \text{as } r \searrow 0, \\ C_8 r^{-2} + o(r^{-2}) & \text{as } r \nearrow \infty, \end{cases}$$

imply that

$$-r\mathbf{B}'(r) \geq C_9 \min\{r^2, r^{-2}\}.$$

Then in view of (2.26), we have for $0 < r \leq 1$,

$$\left| -\lambda\beta + \frac{\lambda+1}{2} r\beta' - \frac{\mathcal{Q}_r[\beta]}{2(n-1)} \right| \leq C_{10} r^2,$$

and hence

$$\begin{aligned} \mathcal{T}_r\left[z^+_{\text{int}}\right] &\geq -e^{-\lambda\tau} r\mathbf{B}'(r) - e^{-2\lambda\tau} C_{10} r^2 \\ &\geq e^{-\lambda\tau} r^2 \left(C_9 - e^{-\lambda\tau} C_{10}\right) \\ &> 0, \end{aligned}$$

for all $\tau \geq \tau_1$ with τ_1 sufficiently large. And for $r \geq 1$,

$$\left| -\lambda\beta + \frac{\lambda+1}{2} r\beta' - \frac{\mathcal{Q}_r[\beta]}{2(n-1)} \right| \leq C_{11},$$

so then

$$\begin{aligned} \mathcal{T}_r\left[z^+_{\text{int}}\right] &\geq -e^{-\lambda\tau} r\mathbf{B}'(r) - e^{-2\lambda\tau} C \\ &\geq e^{-\lambda\tau} \left(C_9 r^{-2} - e^{-\lambda\tau} C_{11}\right) \\ &> 0, \end{aligned}$$

if $r < B_1 e^{\lambda\tau/2}$ with constant $B_1 := \sqrt{C_9/C_{11}}$.

Therefore, z^+_{int} is indeed a supersolution. That z^-_{int} is a subsolution is proved similarly. $\square$

2.4.2 In the exterior region

In the u-coordinate, z evolves by equation (2.10), which we rewrite as $\mathcal{D}_u[z] = 0$, where

$$\mathcal{D}_u[z] := \partial_\tau|_u z - \frac{1}{2}(u^{-1} - u) z_u - u^{-2} z - \frac{\mathcal{Q}_u[z]}{2(n-1)}. \tag{2.27}$$

We call z a subsolution (supersolution) of $\mathcal{D}_u[z]=0$ if $\mathcal{D}_r[z]\leq 0$ (≥ 0).

Lemma 2.5. *Let $\lambda \geq 1$. Define $Z_1(u) := u^{-2}(1-u^2)^{1+\lambda}$. Given $A_2>0$, there exist a function $\zeta:(0,1)\to\mathbb{R}$, a constant $B_2>0$, a sufficiently large $\tau_2<\infty$, and a constant $A_3^*<\infty$ depending only on A_2 such that for any $A_3\geq A_3^*$, the functions*

$$z_{ext}^{\pm}(u,\tau) := e^{-\lambda\tau}A_2 Z_1(u) \pm e^{-2\lambda\tau}A_3\zeta(u) \tag{2.28}$$

are sub- (z_{ext}^-) and super- (z_{ext}^+) solutions of $\mathcal{D}_u[z]=0$ in the exterior region

$$\Omega_{ext} := \left\{ B_2\sqrt{\frac{A_3}{A_2}}e^{-\lambda\tau/2} \leq u < 1 \right\}, \tag{2.29}$$

for $\tau\geq\tau_2$ where τ_2 depends only on A_2 and A_3.

Proof. Since A_2Z_1 is a solution of the ODE (2.17), we have

$$\begin{aligned} e^{2\lambda\tau}\mathcal{D}_u[z_{\text{ext}}^+] = A_3\left\{-\frac{1}{2}(u^{-1}-u)\zeta' - (u^{-2}+2\lambda)\zeta\right\} - \frac{A_2^2}{2(n-1)}\mathcal{Q}_u[Z_1] \\ - \frac{A_2A_3}{n-1}e^{-\lambda\tau}\hat{\mathcal{Q}}_u[Z_1,\zeta] - \frac{A_3^2}{n-1}e^{-2\lambda\tau}\mathcal{Q}_u[\zeta]. \end{aligned}$$

Since $0<u<1$, the definition of Z_1 implies that

$$|Z_1'| \leq \frac{C_1}{u(1-u^2)}Z_1, \quad |Z_1''| \leq \frac{C_2}{u^2(1-u^2)^2}Z_1, \tag{2.30}$$

and from (2.20),

$$\Big|\mathcal{Q}_u[Z_1]\Big| \leq C_3u^{-6}(1-u^2)^{2\lambda}. \tag{2.31}$$

Let $\zeta:(0,1)\to\mathbb{R}$ be a solution of the inhomogeneous ODE

$$-\frac{1}{2}(u^{-1}-u)\zeta' - (u^{-2}+2\lambda)\zeta = u^{-6}(1-u^2)^{2\lambda}. \tag{2.32}$$

Then we solve the ODE to obtain

$$\zeta(u) = u^{-4}(1-u^2)^{2\lambda} h(u),$$

where

$$\begin{aligned} h(u) = {} & 1 - 2u^2 + C_4 u^2 (1-u^2) \\ & + 2u^2(1-u^2)\left[\log(1-u^2) - 2\log u\right] \end{aligned}$$

for an arbitrary constant C_4. This implies that ζ has the asymptotic behavior

$$\zeta(u) = \begin{cases} u^{-4} + O\left(u^{-2}\log u\right) & \text{as } u \searrow 0, \\ -(1-u^2)^{2\lambda} + O\left(\left(1-u^2\right)^{1+2\lambda}\log(1-u^2)\right) & \text{as } u \nearrow 1. \end{cases} \tag{2.33}$$

We then have the following estimates. For $0 < u < 1/2$,

$$\left|\hat{\mathcal{Q}}_u[Z_1, \zeta]\right| \le C_5 u^{-8}, \quad \left|\mathcal{Q}_u[\zeta]\right| \le C_6 u^{-10}. \tag{2.34}$$

For $1/2 \le u < 1$,

$$\left|\hat{\mathcal{Q}}_u[Z_1, \zeta]\right| \le C_7(1-u^2)^{3\lambda-1}, \quad \left|\mathcal{Q}_u\left[\zeta\right]\right| \le C_8(1-u^2)^{4\lambda-2}. \tag{2.35}$$

Using the definition of ζ and the estimate (2.31), we have

$$\begin{aligned} e^{2\lambda\tau}\mathcal{D}_u[z^+_{\text{ext}}] \ge {} & \left(A_3 - C_3 A_2^2\right) u^{-6}(1-u^2)^{2\lambda} \\ & - \frac{A_2 A_3}{n-1} e^{-\lambda\tau}\left|\hat{\mathcal{Q}}_u\left[Z_1, \zeta\right]\right| - \frac{A_3^2}{n-1} e^{-2\lambda\tau}\left|\mathcal{Q}_u\left[\zeta\right]\right|. \end{aligned}$$

We choose $A_3^* = 2C_3 A_2^2$. Then for $A_3 \ge A_3^*$, we have the following. For $0 < u \le 1/2$, there exists a constant $B_2 < \infty$ such that (2.34) implies

$$\begin{aligned} e^{2\lambda\tau}\mathcal{D}_u[z^+_{\text{ext}}] & \ge C_9 u^{-6}\left(A_2^2 - C_5 A_2 A_3 u^{-2} e^{-\lambda\tau} - C_6 A_3^2 u^{-4} e^{-2\lambda\tau}\right) \\ & \ge 0 \end{aligned}$$

for $e^{\lambda\tau}u^2 \geq B_2^2 A_3/A_2$, that is,

$$B_2\sqrt{\frac{A_3}{A_2}}e^{-\lambda\tau/2} \leq u \leq \frac{1}{2}.$$

For $1/2 \leq u < 1$, writing $v := 1-u^2$, then in view of (2.35),

$$\begin{aligned} e^{2\lambda\tau}\mathcal{D}_u[z^+_{\text{ext}}] &\geq C_{10}\Big(A_2^2 - C_7A_2A_3e^{-\lambda\tau}v^{\lambda-1} - C_8A_3^2e^{-2\lambda\tau}v^{2\lambda-2}\Big)v^{2\lambda} \\ &\geq 0 \end{aligned}$$

if $\tau \geq \tau_2$ with τ_2 sufficiently large.

Therefore, z^+_{ext} is indeed a supersolution. That z^-_{ext} is a subsolution is proved similarly. $\square$

2.5 Barriers

Recall the formal solution z_{form} in equation (2.21) that was constructed in Section 2.3. A lower (upper) barrier is a subsolution (supersolution) that lies below (above) z_{form} in an appropriate space-time region. The main result of this section is the following.

Proposition 2.6. *There exist a sufficiently large $\tau_0 < \infty$ and positive piecewise smooth[5] functions $z^\pm(u,\tau)$, $0 < u < 1$ and $\tau \geq \tau_0$, such that the following are true.*

(B1) $z^\pm$ are sub- $(-)$ and super- $(+)$ solutions to equation (2.10).

[5]We note that, cf. Remark 2.12, the assumption of $z^\pm$ being piecewise smooth will still allow us to prove a comparison principle, i.e., Lemma 2.11.

(B2) $z^-(u,\tau_0) < z_{form}(u,\tau_0) < z^+(u,\tau_0)$ *for* $u \in (0,1)$.

(B3) Near $u = 0$, $z^\pm = z^\pm_{int}$; *near* $u = 1$, $z^\pm = z^\pm_{ext}$.

(B4) At any $\tau \in (\tau_0, \infty)$, $\lim\limits_{u\searrow 0} z^- = \lim\limits_{u\searrow 0} z^+ = 1$, *and* $\lim\limits_{u\nearrow 1} z^- = \lim\limits_{u\nearrow 1} z^+ = 0$.

(B5) At any $\tau \in (\tau_0, \infty)$, *there exists a constant* K *independent of* τ *such that*

$$\left|z^\pm_u/u\right|,\ \left|z^\pm_{uu}\right| \le K e^{\lambda\tau}, \tag{2.36}$$

at points where $z^\pm$ *are smooth.*

The proposition will follow from several lemmata. We first explain the idea behind its proof. We properly order $z^\pm_{\text{ext}}$ and $z^\pm_{\text{int}}$ so that $z^-_{\text{int}} \le z^+_{\text{int}}$, $z^-_{\text{ext}} \le z^+_{\text{ext}}$. We then patch together z^+_{int} and z^+_{ext} near the interior-exterior interface to obtain an upper barrier. A similar patching argument yields a lower barrier.

Lemma 2.7. *Let* β *be defined as in Lemma 2.4. Let* A_1^+ *and* A_1^- *denote the constant* A_1 *in* z^+_{int} *and* z^-_{int}, *respectively. For* $A_1^- > A_1^+$, *there exists* $\tau_3 \ge \tau_1$ *such that*

$$z^\pm_{int} = \mathfrak{B}(A_1^\pm r) \pm e^{-\lambda\tau}\beta$$

are properly ordered in Ω_{int} *for all* $\tau \ge \tau_3$.

Proof. For $A_1^- > A_1^+$, using the asymptotic expansions of $\mathfrak{B}$ and β (cf. the proof of Lemma 2.4) we have the following. Near $r = 0$, with $b_2 > 0$,

$$\begin{aligned} z^+_{\text{int}} - z^-_{\text{int}} &= \left\{b_2\left[(A_1^-)^2 - (A_1^+)^2\right] + 2[1+o(1)]e^{-\lambda\tau}\right\} r^2 + O(r^4) \\ &> 0 \quad \text{as } r \searrow 0. \end{aligned}$$

Near $r = \infty$, with $\hat{A} = 1 + \frac{\lambda - 1}{2}$,

$$\begin{aligned} z_{\text{int}}^{+} - z_{\text{int}}^{-} &= \left[(A_1^{+})^{-2} - (A_1^{-})^{-2}\right]\left\{r^{-2} - 2[\hat{A} + o(1)]e^{-\tau}\right\} + O\left(r^{-4}\right) \\ &> 0 \end{aligned}$$

for sufficiently large τ and r. On any bounded interval $c < r < C$ and for sufficiently large τ, it is straightforward to check that $z_{\text{int}}^{+} > z_{\text{int}}^{-}$. Thus, the lemma follows. □

Lemma 2.8. *Let Z_1 and ζ be defined as in Lemma 2.5. Let A_2^{+} and A_2^{-} denote the constant A_2 in z_{ext}^{+} and z_{ext}^{-}, respectively. For $A_2^{+} > A_2^{-}$, if we relabel $A_3 := \max\{A_3(A_2^{+}), A_3(A_2^{-})\}$, $B_2 := \max\{B_2(A_2^{+}), B_2(A_2^{-})\}$, $\tau_2 :=$ $\max\{\tau_2(A_2^{+}), \tau_2(A_2^{-})\}$, then there exists $\tau_4 \geq \tau_2$ such that*

$$z_{ext}^{\pm}(u, \tau) = e^{-\lambda\tau}A_2^{\pm}Z_1(u) \pm e^{-2\lambda\tau}A_3\zeta(u)$$

are properly ordered in ${\Omega_{ext}}$[6] for all $\tau \geq \tau_4$.

Proof. For $A_2^{+} > A_2^{-}$, the asymptotic expansions (2.33) of ζ imply the following. As $u \searrow 0$, $z_{\text{ext}}^{+} > z_{\text{ext}}^{-}$. As $u \nearrow 1$,

$$\begin{aligned} z_{\text{ext}}^{+} - z_{\text{ext}}^{-} &= e^{-\lambda\tau}(1-u^2)^{1+\lambda}\left\{(A_2^{+} - A_2^{-})u^{-2} - 2A_3e^{-\lambda\tau}(1-u^2)^{\lambda-1}\right\} \\ &\quad + e^{-2\lambda\tau}O\left((1-u^2)^{\lambda}\log(1-u^2)\right) \\ &> 0 \end{aligned}$$

[6] In the definition (2.29) of Ω_{ext} we replace A_2 with A_2^{-} since $A_2^{+} > A_2^{-}$.

for all τ sufficiently large. On any interval $0 < a \leq u \leq b < 1$ and for sufficiently large τ, $z^+_{\text{ext}} > z^-_{\text{ext}}$ by a direct computation. Thus, the lemma is proved. $\square$

For sufficiently large τ, Ω_{int} and Ω_{ext} intersect. In below, we state and prove a patching lemma for z^+_{int} and z^+_{ext}. We omit the patching lemma for z^-_{int} and z^-_{ext}, since its statement and proof are entirely analogous. To shorten notations, we write A^+_1 and A^+_2 as A_1 and A_2.

Lemma 2.9. *Let $R_D := D\sqrt{A_3/A_2}$ where $D > 0$ is arbitrary. Suppose A_1 and A_2 satisfy the following inequality*

$$(1 + \frac{3}{8}D^{-2})A_2 < A_1^{-2} < (1 + \frac{1}{2}D^{-2})A_2. \tag{2.37}$$

Then there exists $\tau_5 := \max\{\tau_3, \tau_4\}$ sufficiently large such that

$$z^+_{int} \leq z^+_{ext} \quad at\ r = R_D, \tag{2.38}$$

$$z^+_{int} \geq z^+_{ext} \quad at\ r = 2R_D, \tag{2.39}$$

for $\tau \geq \tau_5$.

Proof. At the interface of the interior and the exterior regions, we have the following when $\tau \geq \tau_5$. From the interior region, as $r \nearrow \infty$, $\mathfrak{B}(r) = r^{-2} + c_2 r^{-4} + O(r^{-6})$, and so

$$z^+_{\text{int}} = A_1^{-2} r^{-2} + c_2 A_1^{-4} r^{-4} + O(r^{-6}) + O(e^{-\lambda\tau}) \quad \text{as } r \nearrow \infty.$$

From the exterior region, as $u \searrow 0$, using $u = re^{-\lambda\tau/2}$ and (2.33), we have on any compact r-interval,

$$
\begin{aligned}
z_{\text{ext}}^{+} &= A_2 e^{-\lambda\tau} u^{-2} (1-u^2)^{\lambda+1} + A_3 e^{-2\lambda\tau} u^{-4} \left(1 + O\left(u^2 \log u\right)\right) \\
&= A_2 r^{-2} + A_3 r^{-4} + O(\tau e^{-\lambda\tau}).
\end{aligned}
$$

Then on bounded r-interval, one has

$$
r^2 \left(z_{\text{int}}^{+} - z_{\text{ext}}^{+}\right) = (A_1^{-2} - A_2) + \left(c_2 A_1^{-4} + O(r^{-2}) - A_3\right) r^{-2} + O(\tau e^{-\lambda\tau}).
$$

We can choose a constant $\hat{C}$ so large that for

$$
A_3 \geq \hat{C} A_1^{-4} \quad \text{and} \quad A_3 \geq \hat{C}\sqrt{A_2},
$$

we have

$$
\left| \frac{c_2 A_2}{A_3 A_1^4} + O\left(\frac{A_2^2}{A_3^2}\right) \right| \leq \frac{A_2}{2}.
$$

Then at $r = R_D$,

$$
\begin{aligned}
R_D^2 \left(z_{\text{int}}^{+} - z_{\text{ext}}^{+}\right) &= (A_1^{-2} - A_2) + \left[\frac{c_2 A_2}{A_3 A_1^4} + O\left(\frac{A_2^2}{A_3^2}\right) - A_2 \right] D^{-2} + O(\tau e^{-\lambda\tau}) \\
&\leq A_1^{-2} - \left(1 + \frac{1}{2} D^{-2}\right) A_2 + O(\tau e^{-\lambda\tau}),
\end{aligned}
$$

and at $r = 2R_D$,

$$
\begin{aligned}
4R_D^2 \left(z_{\text{int}}^{+} - z_{\text{ext}}^{+}\right) &= (A_1^{-2} - A_2) + \left[\frac{c_2 A_2}{A_3 A_1^4} + O\left(\frac{A_2^2}{A_3^2}\right) - A_2 \right] \frac{D^{-2}}{4} + O(\tau e^{-\lambda\tau}) \\
&\geq A_1^{-2} - \left(1 + \frac{3}{8} D^{-2}\right) A_2 + O(\tau e^{-\lambda\tau}).
\end{aligned}
$$

Now choose A_1 and A_2 according to (2.37), then the lemma follows for $\tau \geq \tau_5$. $\square$

Lemmata 2.7, 2.8, and 2.9 allow us to construct barriers for equation (2.10). From now on, we denote by z^+ a function defined by

$$z^+(u,\tau) := \begin{cases} z^+_{\text{int}}, & \text{if} \quad 0 < u \le e^{-\lambda\tau/2}R_D, \\ \min\{z^+_{\text{int}}, z^+_{\text{ext}}\}, & \text{if} \quad e^{-\lambda\tau/2}R_D \le u \le 2e^{-\lambda\tau/2}R_D, \\ z^+_{\text{ext}}, & \text{if} \quad 2e^{-\lambda\tau/2}R_D \le u < 1, \end{cases} \tag{2.40}$$

for $\tau \ge \tau_5$. We define z^- analogously using z^-_{int} and z^-_{ext}. In particular, for $e^{-\lambda\tau/2}R_D \le u \le 2e^{-\lambda\tau/2}R_D$, $z^- = \max\{z^-_{\text{int}}, z^-_{\text{ext}}\}$.

Lemma 2.10. *Let $\tau \in (\tau_5, \infty)$. There exists a constant K independent of τ such that*

$$\left|z^\pm_u/u\right|,\ \left|z^\pm_{uu}\right| \le Ke^{\lambda\tau}$$

at points where $z^\pm$ are smooth.

Proof. At a point where z^+ is smooth, z^+ is either z^+_{int} or z^+_{ext}.

Suppose z^+ is smooth at $u \in (0, 2e^{-\lambda\tau/2}R_D)$ and $z^+ = z^+_{\text{int}}$, then

$$\begin{aligned} z^+ &= \mathfrak{B}(A_1 r) + e^{-\lambda\tau}\beta(r) \\ &= 1 + C_1 r^2 + o(r^2) + e^{-\lambda\tau}\left(r^2 + o(r^2)\right) \quad \text{as } r \searrow 0, \\ &= 1 + C_1 e^{\lambda\tau}u^2 + e^{\lambda\tau}o(u^2) + u^2 + o(u^2) \quad \text{as } u \searrow 0. \end{aligned}$$

So then

$$\begin{aligned} z^+_u &= e^{\lambda\tau}\left(C_2 u + o(u)\right) + u + o(u) \quad \text{as } u \searrow 0, \\ z^+_{uu} &= e^{\lambda\tau}\left(C_3 + o(1)\right) + 1 + (o(1) \quad \text{as } u \searrow 0. \end{aligned}$$

Thus, there exists a constant K_1 such that for $0 < u < 2e^{-\lambda\tau/2} R_D$,

$$|z_u^+/u|,\ |z_{uu}^+| \leq K_1 e^{\lambda\tau}. \tag{2.41}$$

Suppose z^+ is smooth at $u \in (e^{-\lambda\tau/2} R_D, 1)$ and $z^+ = z^+_{\text{ext}}$, then

$$z^+ = e^{-\lambda\tau} A_2 Z_1(u) + e^{-2\lambda\tau} A_3 \zeta(u),$$

where $Z_1(u) = u^{-2}(1-u^2)^{\lambda+1}$ for $\lambda \geq 1$, and $\zeta(u)$ is a smooth solution to the ODE (2.32). So then

$$\begin{aligned} |z_u^+/u| &\lesssim e^{-\lambda\tau}|Z_1'/u| + e^{-2\lambda\tau}|\zeta'/u|, \\ |z_{uu}^+| &\lesssim e^{-\lambda\tau}|Z_1''| + e^{-2\lambda\tau}|\zeta''|. \end{aligned}$$

From the definition of Z_1, we compute

$$Z_1'/u = -2\left(u^{-4} + \lambda u^{-2}\right)(1-u^2)^{\lambda}, \tag{2.42}$$

$$Z_1'' = 2(1-u^2)^{\lambda-1}\left[3u^{-4} + 3(\lambda-1)u^{-2} + (2\lambda-1)\lambda\right]. \tag{2.43}$$

From equation (2.32), we have

$$-\frac{1}{2}\zeta' = \frac{(1-u^2)^{2\lambda-1}}{u^5} + \frac{u^{-1}+\lambda u}{(1-u^2)}\zeta.$$

Then using (2.33) we obtain, writing $v := 1-u^2$,

$$|\zeta'/u| \lesssim \begin{cases} u^{-6} + O\left(u^{-4}\log u\right) & \text{as } u \searrow 0, \\ v^{2\lambda-1} + O\left(v^{2\lambda}\log v\right) & \text{as } u \nearrow 1. \end{cases} \tag{2.44}$$

and similarly,

$$|\zeta''| \lesssim \begin{cases} u^{-6} + O\left(u^{-4}\log u\right) & \text{as } u \searrow 0, \\ v^{2(\lambda-1)} + O\left(v^{2\lambda-1}\log v\right) & \text{as } u \nearrow 1. \end{cases} \tag{2.45}$$

Thus, by (2.42)–(2.45), there exist constants K_2, K_3 such that for $e^{-\lambda\tau/2}R_D < u < 1$,

$$|z_u^+/u| \le K_2 e^{\lambda\tau}, \quad |z_{uu}^+| \le K_3 e^{\lambda\tau}.$$

Choose $K = \max\{K_1, K_2, K_3\}$, then the lemma is true for z^+. The proof for z^- is similar. $\square$

We can now prove Proposition 2.6.

Proof of Proposition 2.6. Since $\lim_{u \searrow 0} z_{\text{int}}^{\pm} = 1$, $z_{\text{int}}^{\pm} > 0$ on $0 < r \le 2R_D$ for sufficiently small D. Since $Z_1(u) \ge 0$, there exists a sufficiently large $\tau_0 \ge \tau_5$ such that $z_{\text{ext}}^{\pm} > 0$ on $e^{-\lambda\tau/2}R_D \le u < 1$. Thus, $z^{\pm}$ are positive piecewise smooth functions for $0 < u < 1$ and $\tau \ge \tau_0$. The minimum (maximum) of two supersolutions (subsolutions) is still a supersolution (subsolution), so (B1) is true. One checks (B2)–(B4) directly using the definition of $z^{\pm}$ and the properties of $z_{\text{int}}^{\pm}$ and $z_{\text{ext}}^{\pm}$. (B5) follows from Lemma 2.10. $\square$

2.6 Existence and uniqueness of complete solutions

We first prove a comparison principle for equation (2.10). Similar results have appeared in [4, 66].

Lemma 2.11. *Let $\bar{\tau} \in [\tau_0, \infty)$ be arbitrary. Let $z^{\pm}$ be two nonnegative sub- $(-)$ and super- $(+)$ solutions (of equation* (2.10) *respectively. Suppose there exists a constant K such that either $|z_u^-/u|$, $|z_{uu}^-|$ or $|z_u^+/u|$, $|z_{uu}^+|$ are bounded by $Ke^{\lambda\tau}$. Moreover, assume*

(C1) $z^-(u,\tau_0) < z^+(u,\tau_0)$ *for* $0 < u < 1$*;*

(C2) $z^-(0,\tau) \le z^+(0,\tau)$*, and* $z^-(1,\tau) \le z^+(1,\tau)$ *for all* $\tau \in [\tau_0, \bar{\tau}]$*.*

Then $z^-(u,\tau) \le z^+(u,\tau)$ *in* $[0,1] \times [\tau_0, \bar{\tau}]$*.*

Remark 2.12. *In this lemma, we assume* $z^\pm$ *are smooth. The result holds for piecewise smooth* $z^\pm$*. When applying the comparison principle, we will only evaluate* $z^\pm$ *at "points of first contact with a given smooth function" which are necessarily smooth points of* $z^\pm$ *for each* $\tau \ge \tau_0$*.*

Proof of Lemma 2.11. Suppose $|z_u^+/u|, |z_{uu}^+| \le Ke^{\lambda\tau}$. For $\mu > 0$ to be chosen and arbitrary $\varepsilon > 0$, define a function

$$w := e^{-\mu e^{\lambda\tau}}(z^+ - z^-) + \varepsilon.$$

Then $w > 0$ on the parabolic boundary of the evolution by assumptions (C1) and (C2). We claim that $w > 0$ in $(0,1) \times [\tau_0, \bar{\tau}]$. Suppose the contrary, there must be an interior point u_* and a first time τ_* such that $w(u_*, \tau_*) = 0$ and $w_\tau(u_*, \tau_*) \le 0$. Moreover, at (u_*, τ_*), we have

$$z^+ = z^- - \varepsilon e^{-\mu e^{\lambda\tau_*}}, \quad z_u^+ = z_u^-, \quad z_{uu}^+ \ge z_{uu}^-.$$

Then at (u_*, τ_*),

$$\begin{aligned}
0 \geq e^{\mu e^{\lambda\tau_*}} w_\tau &= \left(z^+_\tau - z^-_\tau\right) - \lambda\mu e^{\lambda\tau_*}\left(z^+ - z^-\right) \\
&= \left(z^+ - z^-\right)\left(u^{-2} - \lambda\mu e^{\lambda\tau_*}\right) + \frac{\mathcal{Q}_u[z^+] - \mathcal{Q}_u[z^-]}{2(n-1)} \\
&= \left(z^- - z^+\right)\left\{\lambda\mu e^{\lambda\tau_*} + \frac{(z^+_u/u) - z^+_{uu}}{2(n-1)} + \frac{z^+ + z^- - 1}{u^2}\right\} \\
&\quad + z^-\left(z^+_{uu} - z^-_{uu}\right) \\
&\geq \varepsilon e^{-\mu e^{\lambda\tau_*}}\left\{\lambda\mu e^{\lambda\tau_*} - \frac{Ke^{\lambda\tau_*}}{(n-1)} - \frac{1}{u_*^2}\right\}.
\end{aligned}$$

Choose μ so large that $\lambda\mu > K/(n-1) + u_*^{-2}e^{-\lambda\tau_*}$, then at (u_*, τ_*),

$$0 \geq w_\tau > 0,$$

which is a contradiction. This proves the case $|z^+_u/u|, |z^+_{uu}| \leq Ke^\tau$.

The case when $|z^-_u/u|, |z^-_{uu}| \leq Ke^{\lambda\tau}$ is proved analogously because at the interior first contact point (u_*, τ_*), we have

$$\begin{aligned}
e^{\mu e^{\lambda\tau_*}} w_\tau &= \left(z^- - z^+\right)\left\{\mu\lambda e^{\lambda\tau_*} + \frac{(z^-_u/u) - z^-_{uu}}{2(n-1)} + \frac{z^+ + z^- - 1}{u^2}\right\} \\
&\quad + z^+\left(z^+_{uu} - z^-_{uu}\right).
\end{aligned}$$

Therefore, the lemma is proved. □

Now for any solution z of equation (2.10) we have the following.

Lemma 2.13. *Suppose* $0 < z \leq z^+$, *then* z *defines a* complete *rotationally symmetric metric* $g := z^{-1}d\psi^2 + \psi^2 g_{sph}$ *on* $\mathbb{R}^{n+1}$.

Proof. By definition g is rotationally symmetric. To see g is a complete metric, it suffices to show that any radial geodesic η starting from the origin has infinite length in the s-coordinate. The length of η in s-coordinate is a function of u and τ given by

$$s(u,\tau) = e^{-\tau/2}\sigma(u) = e^{-\tau/2}\int_0^u \frac{d\sigma}{d\hat{u}}d\hat{u}.$$

Since $z = \psi_s^2 = 2(n-1)u_\sigma^2$, and $0 < z \leq z^+$ by hypothesis, we have

$$\frac{\sigma(u)}{\sqrt{2(n-1)}} \geq \int_{u_0}^u \frac{1}{\sqrt{z}}d\hat{u} \geq \int_{u_0}^u \frac{1}{\sqrt{z^+}}d\hat{u}.$$

As $u \nearrow 1$,

$$\begin{aligned}
z^+_{\text{ext}} &= e^{-\lambda\tau}A_2u^{-2}(1-u^2)^{1+\lambda} + e^{-2\lambda\tau}A_3\zeta(u),\\
&= e^{-\lambda\tau}A_2u^{-2}(1-u^2)^{\lambda+1}\\
&\quad + e^{-2\lambda\tau}A_3\left\{-(1-u^2)^{2\lambda} + O\left((1-u^2)^{2\lambda+1}\log(1-u^2)\right)\right\}.
\end{aligned}$$

So for u_0 and τ_0 sufficiently large, $z^+ = z^+_{\text{ext}}$ in $[u_0,1)\times[\tau_0,\infty)$ with

$$z^+_{\text{ext}} \leq e^{-\lambda\tau}u^{-2}(1-u^2)^{1+\lambda}\left(\frac{3A_2}{2}\right).$$

It follows that

$$\begin{aligned}
\frac{s(u,\tau)}{\sqrt{2(n-1)}} &\geq e^{-\tau/2}\int_{u_0}^u \frac{1}{\sqrt{z^+}}d\hat{u} = e^{-\tau/2}\int_{u_0}^u \frac{1}{\sqrt{z^+_{\text{ext}}}}d\hat{u}\\
&\geq \sqrt{\frac{3A_2}{2}}e^{\lambda\tau/2}\int_{u_0}^u \frac{\hat{u}}{(1-\hat{u}^2)^{(1+\lambda)/2}}d\hat{u}.
\end{aligned}$$

Hence,

$$\frac{s(u,\tau)}{\sqrt{3(n-1)A_2}} \geq \begin{cases} \log(1-u_0^2) - \log(1-u^2), & \lambda = 1,\\ \frac{e^{(\lambda-1)\tau/2}}{(\lambda-1)}\left\{(1-u^2)^{(1-\lambda)/2} - (1-u_0^2)^{(1-\lambda)/2}\right\}, & \lambda > 1. \end{cases}$$

Therefore, for each $\tau \geq \tau_0$, $\lim\limits_{u\nearrow 1} s(u,\tau) = \infty$, thus proving the lemma. □

We are now ready to prove the main result of this chapter.

Proof of Theorem 2.1. Let $\hat{z}_0$ be the function obtained by patching together $\mathfrak{B}(\tilde{A}_1 r)$ and $\tilde{A}_2 Z_1(u)$, where $A_1^+ < \tilde{A}_1 < A_1^-$ and $A_2^- < \tilde{A}_1 < A_2^+$. Because $z^-(u,\tau_0) < z^+(u,\tau_0)$, we can smooth out $\hat{z}_0$ to obtain a smooth initial profile z_0 with $0 < z^-(u,\tau_0) < z_0 < z^+(u,\tau_0)$ for $0 < u < 1$. By Lemma 2.13, z_0 determines a complete rotationally symmetric metric g^0 on $\mathbb{R}^{n+1}$. It is straightforward to check that g^0 has bounded sectional curvatures. Since the sectional curvatures depend smoothly on the metric, there is a neighborhood $\mathcal{G}_{n+1}$ of g^0 in the C^2-topology that corresponds to an open set of z all of which lie between $z^-(u,\tau_0)$ and $z^+(u,\tau_0)$.

Let $g_0 \in \mathcal{G}_{n+1}$. There exists a unique solution $g(t)$ to Ricci flow for $t \in [0,T_0)$ with $g(0) = g_0$ [75, 20]. By expression (2.1), g_0 has a ψ-profile function $\psi(s,0) < r_0$ for some constant $r_0 > 0$. Also, the metric $\tilde{g}_t = ds^2 + \tilde{\psi}(t)^2 g_{\text{sph}}$ with $\tilde{\psi}(0) \equiv r_0$ is a shrinking cylinder solution to Ricci flow on $\mathbb{R} \times S^n$. Under Ricci flow, ψ evolves by the equation

$$\partial_t \psi = \psi_{ss} - (n-1)\frac{1-\psi_s^2}{\psi},$$

whereas $\tilde{\psi}$ evolves by the equation

$$\partial_t \tilde{\psi} = -(n-1)/\tilde{\psi}.$$

So we have

$$\partial_t(\psi - \tilde{\psi}) = (\psi - \tilde{\psi})_{ss} + (n-1)\frac{(\psi - \tilde{\psi})_s^2 + (\psi - \tilde{\psi})}{\psi\tilde{\psi}},$$

where $\psi, \tilde{\psi} > 0$ for $s > 0$ before the first singular time T. Then it follows from the maximum principle that $\psi(t) \leq \tilde{\psi}(t)$. In particular, $\tilde{\psi}(t) = 0$ at time $\tilde{T} = \frac{r_0^2}{2(n-1)}$. Hence, $g(t)$ encounters a global singularity in finite time $T \leq \frac{r_0^2}{2(n-1)}$.

The profile $z(u, \tau)$ of $g(t)$ is the unique solution of equation (2.10) for $0 < u < 1$ and $\tau \geq \tau_0$, with boundary conditions $z(0, \tau) = 1$ and $z(1, \tau) = 0$, and initial condition $z(u, \tau_0) = z_0$. The barriers $z^{\pm}$ satisfy the hypotheses of Lemma 2.11, so $z^- \leq z(u, \tau) \leq z^+$ by the comparison principle for $\tau_0 \leq \tau < \infty$. So for $0 \leq t < T = e^{-\tau_0}$, $g(t)$ corresponding to $z(u, \tau)$ is a complete metric on $\mathbb{R}^{n+1}$ by Lemma 2.13.

The sectional curvatures of $g(t)$ at the origin O are

$$K|_O = L|_O = \lim_{x \searrow 0} \frac{1 - \psi_s^2}{\psi^2} = \lim_{r \searrow 0} \frac{1 - z}{r^2} e^{2\lambda\tau} = \frac{C}{(T - t)^{2\lambda}},$$

where C is a positive constant depending on n. So part (1) of Theorem 2.1 is proved.

Since $z^- \leq z(u, \tau) \leq z^+$ for any $\tau < \infty$, the solution $z(u, \tau)$ exhibits the asymptotic behavior of $z^{\pm}$. Near the origin, $z(u, \tau)$ converges uniformly to the Bryant soliton profile function for $0 < u < R_D e^{-\lambda\tau}$ and $\tau \nearrow \infty$. Near spatial infinity, $u \nearrow 1$ while $z(u, \tau) \searrow 0$. Thus, $g(t)$ has asymptotic behavior described in parts (2) and (3) of Theorem 2.1. $\square$

2.7 Relation to the standard solutions

In [71], Perelman described a special family of Ricci flow solutions, the so-called standard solutions, on $\mathbb{R}^3$. These solutions are complete rotationally symmetric with nonnegative sectional curvature, and split at infinity as the metric product of a ray and the round 2-sphere of constant scalar curvature.

Consider a rotationally symmetric metric g_0 on $\mathbb{R}^{n+1}$ with the following properties:

(P1) $\mathrm{Rm}_{g_0} \geq 0$ everywhere with $\mathrm{Rm}_{g_0} > 0$ at some point.

(P2) The curvature $|\mathrm{Rm}_{g_0}|$ and its derivatives $|\nabla^i \mathrm{Rm}_{g_0}|$, $i = 1, 2, 3, 4$ are bounded.

(P3) There is a sequence of points $y_k \to \infty$ in $\mathbb{R}^{n+1}$ such that $(\mathbb{R}^{n+1}, g_0, y_k)$ converges to $\mathbb{R} \times S^n(r_0)$, where $r_0 > 0$ is some constant, in pointed C^3 Cheeger-Gromov topology.

Following [63], a Ricci flow solution $g(t)$ whose initial condition satisfies (P1)–(P3) is called a standard solution. A standard solution of Ricci flow is uniquely determined by its initial datum up to the first singular time [63, 20].

Lemma 2.14. *Let $\mathcal{G}_{n+1}$ be as in Theorem 2.1. There is an open set $\mathcal{G}^*_{n+1} \subset \mathcal{G}_{n+1}$ of metrics that satisfy properties (P1)–(P3).*

Proof. Define

$$\mathcal{G}^*_{n+1} := \{g_0 \in \mathcal{G}_{n+1} : g_0 \text{ satisfies P(1)–P(3)}\}\,.$$

We first show $\mathcal{G}^*_{n+1}$ is nonempty.

Let $\tau = \tau_0$ correspond to $t = 0$. By the proof of Theorem 2.1, there exists $\hat{z}_0$ which is obtained by patching scaled copies of $\mathfrak{B}$ and Z_1. Let $\hat{g}_0$ be the metric determined by the profile function $\hat{z}_0$. For $\hat{g}_0$, $K = -(z_u/u)e^{\lambda\tau_0} = -(z_r/r)e^{2\lambda\tau_0} > 0$ at the origin. Observe that the patching occurs in $R_D \leq r \leq 2R_D$, where $R_D = D\sqrt{A_3/A_2}$ for an arbitrary constant $D > 0$. So by the continuity of K there exists D_0 such that $K > 0$ for $0 < r \leq 2R_0$, where $R_0 := R_{D_0}$. On the other hand, where $\hat{z}_0 = A_2u^{-2}(1-u^2)^{1+\lambda}$, we have

$$K = 2A_2u^{-4}(1-u^2)^{\lambda}(1+\lambda u^2)e^{\lambda\tau} > 0.$$

Hence, the piecewise smooth function $\hat{z}_0$ determines a metric $\hat{g}_0$ for which $K > 0$ in the interior of $\mathbb{R}^{n+1}$ where $\hat{g}_0$ is smooth, and $K \searrow 0$ as $u \nearrow 1$, i.e., as one approaches spatial inifinity. Since $z^- < z^+$, we can smooth $\hat{z}_0$ to obtain a smooth metric g_0 for which $K \geq 0$ everywhere with $K > 0$ at the origin, and $g_0 \in \mathcal{G}_{n+1}$. Also for this metric g_0, because $L = (1-z)/\psi^2$, $L \geq 0$ everywhere with $L > 0$ at the origin, and $L \to 1/\psi^2$ as we approach spatial infinity. Thus, g_0 satisfies (P1).

To check (P2), we first note that $|\operatorname{Rm}_{g_0}|$ is bounded by the proof of Theorem 2.1. The derivatives $\nabla^i \operatorname{Rm}_{g_0}$, $i \in \mathbb{N}$, are determined by $\partial_s^i K$ and $\partial_s^i L$. Recall that $s(u,\tau) = e^{-\tau/2}\sigma(u)$ and $z = 2(n-1)u_\sigma^2$, then at time τ_0,

$$\frac{\partial s}{\partial u} = \frac{\partial \sigma}{\partial u}e^{-\tau_0/2} = e^{-\tau_0/2}\frac{\sqrt{2(n-1)}}{\sqrt{z_0}}.$$

Since $0 < z^- < z_0 < z^+$, arguing as in the proof of Lemma 2.13, there exists $u_0 \in (0,1)$ such that for $u_0 \le u < 1$,

$$\frac{\partial u}{\partial s} \lesssim \sqrt{z^+_{\text{ext}}} \lesssim (1-u^2)^{(\lambda+1)/2}, \quad \lambda \ge 1. \tag{2.46}$$

By the chain rule that $\partial_s = (\partial u/\partial s)\partial_u$, one checks that

$$K_s \lesssim (1-u^2)^{(3\lambda+1)/2}, \quad L_s \lesssim (1-u^2)^{(\lambda+1)/2} + O\left((1-u^2)^{(3\lambda+1)/2}\right).$$

So K_s and L_s are bounded. Similarly, direct computation shows that $\partial_s^i K$ and $\partial_s^i L$ are bounded for $i = 2, 3, 4$. If $0 < u \le u_0$, then we are looking at a compact subset of $\mathbb{R}^{n+1}$ where $|\nabla^i \operatorname{Rm}_{g_0}|$ are bounded for any $i \in \mathbb{N}$ because g_0 is smooth. Thus, g_0 satisfies (P2).

To check (P3), we let y_k to be a sequence of points whose s-coordinates $s_k \nearrow \infty$ as $k \nearrow \infty$. Let $U_k := (-k, \infty) \times S^n(r_0)$ be an exhaustion of the cylinder $\mathbb{R} \times S^n(r_0)$. Then the translation map $s \mapsto (s + 2k)$ defines an embedding $\psi_k : U_k \to \mathbb{R}^{n+1}$, $V_k := \psi_k(U_k) = (k, \infty) \times S^n(r_0)$. We need to show for $g_0 = ds^2 + \psi(s, \tau_0)^2 g_{\text{sph}}$,

$$g_0|_{V_k} \xrightarrow{C^3} g_{\text{cyl}} \text{ on compact subsets of } \mathbb{R} \times S^n(r_0), \tag{2.47}$$

where $g_{\text{cyl}} = ds^2 + r_0^2 g_{\text{sph}}$ is the standard metric on the round cylinder. Without loss of generality, assume $r_0 = 1$. For all sufficiently large k, the u-coordinate of y_k is bounded between u_0 and 1. At initial time, $\psi \lesssim u$, so $\partial_s^i \psi \lesssim \partial_s^i u$, $i \in \mathbb{N}$. Then at $\tau = \tau_0$, as $s_k \nearrow \infty$, $\psi \lesssim u \nearrow 1$, and hence from (2.46), we obtain

$$\psi_s \lesssim u_s \lesssim (1-u^2)^{\frac{(\lambda+1)}{2}} \searrow 0,$$
$$\psi_{ss} \lesssim u_{ss} \lesssim (1-u^2)^{\lambda} \searrow 0,$$
$$\psi_{sss} \lesssim u_{sss} \lesssim (1-u^2)^{\frac{(3\lambda-1)}{2}} \searrow 0.$$

This shows (2.47)[7], and hence g_0 satisfies (P3).

Therefore, $g_0 \in \mathcal{G}^*_{n+1}$. Since the sectional curvatures depend smoothly on the metric, there is an open set $\mathcal{G}^*_{n+1}$ of g_0 in C^6-topology such that any $g \in \mathcal{G}^*_{n+1}$ satisfies P(1)–P(3). Hence, the lemma follows. □

We now prove Theorem 2.2.

Proof of Theorem 2.2. By Lemma 2.14, the Ricci flow solution $g(t)$ on $\mathbb{R}^{n+1}$ starting at $g_0 \in \mathcal{G}^*_{n+1}$ is a standard solution. Since $g_0 \in \mathcal{G}_{n+1}$, Theorem 2.1 applies to $g(t)$, and so Theorem 2.2 follows. □

[7] One checks that $\partial_s^i \psi \lesssim \partial_s^i u \lesssim (1-u^2)^{1+i\frac{(\lambda-1)}{2}} \searrow 0$ as $u \nearrow 1$, so we in fact have convergence in pointed C^∞ Cheeger-Gromov topology.

Chapter 3

Stability of complex hyperbolic space under curvature-normalized Ricci flow

3.1 Introduction

The linear and dynamical stability of compact flat and Ricci-flat solutions of Ricci flow were first studied by Guenther, Isenberg, and Knopf [41]. They obtained the linear stability of Ricci-flat metrics, i.e., the spectrum of the elliptic differential operator corresponding to the linearized Ricci flow equation at a Ricci-flat metric is contained in $(-\infty, 0]$. Then using the maximal regularity theory developed by Da Prato and Grisvard [29], they concluded the presence of a center manifold in the space of Riemannian metrics and the dynamical stability for any metric whose Ricci flow converges to a flat metric.

The linear and dynamical stability of Ricci flow solutions and Ricci solitons have been analyzed in other contexts. Guenther, Isenberg, and Knopf [42] proved linear stability of homogeneous Ricci solitons, but they fell short of establishing the dynamical stability. The convergence and stability of locally $\mathbb{R}^N$-invariant solutions of Ricci flow was obtained by Knopf [55]. Williams [80] generalized Knopf's results on the volume-rescaled $\mathbb{R}^N$-locally invariant solutions and the curvature-normalized Ricci flow. Very recently, Williams

[81] also studied linear and dynamical stability of solutions of certain extended Ricci flow systems.

In [74], Šešum strengthened the results of [41]. In particular, she proved that the *variational* stability of Ricci flow, which is defined by the nonpositivity of the second variation of Perelman's $\mathcal{F}$-functional, together with an integrability condition, imply the dynamical stability. As a consequence, she obtained the dynamical stability of Ricci-flat metrics and Kähler Ricci-flat metrics. Dai, Wang, and Wei [30] showed that Kähler-Einstein metrics[1] with non-positive scalar curvature are stable as the critical points of the total scalar curvature functional (stable in the sense that the second variation of the functional is nonnegative). Combining their results with Šešum's theorem, the authors established the dynamical stability of compact Kähler-Einstein manifolds with non-positive scalar curvature [30].

The stability question has also been addressed for normalized Ricci flow. Ye [85] proved that in real dimension $n \geq 3$, if the metric has nonzero sectional curvature, then closed Ricci-pinched solutions (defined in [85]) of the volume-normalized Ricci flow converge to an Einstein metric. Li and Yin [61] obtained stability of the hyperbolic metric on $\mathbb{H}^n$ under the curvature-normalized Ricci flow in dimension $n \geq 6$, assuming that the perturbation

[1] On an even-dimensional manifold M, an automorphism $J : TM \to TM$ is called an almost complex structure if $J^2 = -\mathrm{id}_{TM}$. A Riemannian manifold (M, g) is a Kähler manifold if the metric g is J-invariant: $g(JX, JY) = g(X, Y)$, and J is parallel: $\nabla J = 0$. In this case, the metric g is called a Kähler metric. If the metric g is also Einstein, then it is called a Kähler-Einstein metric.

is small and decays sufficiently fast at spatial infinity. Schnürer, Schulze, and Simon [73] established stability of the hyperbolic metric on $\mathbb{H}^n$ under the scaled Ricci harmonic map heat flow in dimension $n \geq 4$. The proofs in these papers rely on showing that a perturbed solution converges exponentially fast to a stationary solution in the L^2-norm.

More recently, Bamler [9] proved that every finite volume hyperbolic manifold (possibly with cusps) in dimension $n \geq 3$ is stable under the curvature-normalized Ricci flow. Later, Bamler [10] obtained more general stability results for symmetric spaces of noncompact type under the curvature-normalized Ricci flow. His results make use of an improved L^1-decay estimate for the heat kernel in vector bundles as well as elementary geometry of negatively curved spaces.

In this chapter, we study the stability of the Bergman metric on complex hyperbolic space under the curvature-normalized Ricci flow following the approach in [41, 42, 55]. We refer the interested reader to [55, Section 2] for a detailed introduction on the maximal regularity theory. For our purpose, we apply the theory in three steps:

1. We choose to study the curvature-normalized Ricci flow so that the Bergman metric is a fixed point of the normalized flow.

2. We linearize the curvature-normalized Ricci flow at the Bergman metric, and study the spectrum of the elliptic operator in the linearized equation.

3. We set up Banach spaces of tensor fields with good interpolation properties, and apply Simonett's Stability Theorem, cf. Theorem 3.3.

Let $(\mathbb{CH}^m, g_B)$ denote the complete noncompact complex m-dimensional hyperbolic space equipped with the Bergman metric g_B. g_B is negatively curved Kähler-Einstein, i.e., $\mathrm{Ric}(g_B) = -\lambda g_B$ for some constant $\lambda > 0$, and g_B has constant holomorphic sectional curvature $-c$ $(c > 0)$ [39]. In this chapter, (M^n, g_0) will denote a smooth closed quotient of $(\mathbb{CH}^m, g_B)$. M^n has real dimension $n = 2m$.

Let $\mathbb{N}$ be the set of positive integers. We now state our main results with respect to the spaces introduced in Section 3.4.

Theorem 3.1. *Let $m \in \mathbb{N}$, $m \geq 2$, and fix $\tau > m/2$. For each $\rho \in (0,1)$, there exists $\eta \in (\rho, 1)$ such that the following is true for $(\mathbb{CH}^m, g_B)$.*

There exists a neighborhood $\mathcal{U}$ of g_B in the $\mathfrak{h}^{1+\eta}_\tau$-topology such that for all initial data $\tilde{g}(0) \in \mathcal{U}$, the unique solution $\tilde{g}(t)$ of the curvature-normalized Ricci-DeTurck flow (3.5) *exists for all $t \geq 0$ and converges exponentially fast in the $\mathfrak{h}^{2+\rho}_\tau$-norm to g_B.*

Theorem 3.2. *Let $m \in \mathbb{N}$, $m \geq 2$, and $n = 2m$. For each $\rho \in (0,1)$, there exists $\eta \in (\rho, 1)$ such that the following is true for (M^n, g_0).*

There exists a neighborhood $\mathcal{U}$ of g_0 in the $\mathfrak{h}^{1+\eta}$-topology such that for all initial data $\tilde{g}(0) \in \mathcal{U}$, the unique solution $\tilde{g}(t)$ of the curvature-normalized Ricci flow (3.4) *exists for all $t \geq 0$ and converges exponentially fast in the $\mathfrak{h}^{2+\rho}$-norm to g_0.*

We note that Theorem 3.2 is a slight improvement of [30, Theorem 1.6], which has convergence in the C^k-norm ($k \geq 3$) for initial data close in the C^k-topology as in [74]. Our method takes optimal advantage of the smoothing properties of the underlying quasilinear parabolic operator of Ricci flow in continuous interpolation spaces. The results in [30] and [74] are more general and are obtained by different techniques.

One may also compare Theorem 3.1 with the $\mathbb{CH}^m$ case in [10, Theorem 1.2], which imposes stronger assumptions on the initial metric and weaker assumptions on the perburbations at spatial infinity, and proves convergence of the Ricci flow solutions to the fixed point in the pointed Cheeger-Gromov sense. In constrast, our initial metric is close to the fixed point in a less regular topology, and we impose stronger assumptions on the perturbations at spatial infinity to prove convergence in a considerably stronger sense.

In order to apply Simonett's Theorem, we define suitably weighted little Hölder spaces $\mathfrak{h}_\tau^{k+\alpha}$ on the complete noncompact $\mathbb{CH}^m$, and establish the interpolation properties of these weighted spaces in all complex dimensions, cf. Theorem 3.4. We remark that the idea of defining weighted spaces is natural when the underlying manifold is complete noncompact and the growth rate of the volume of geodesic balls on this manifold can be estimated from above. If these weighted spaces also satisfy interpolation properties, then the maximal regularity theory becomes applicable. For example, this approach works on $\mathbb{R}^n$ because the interpolation theory on $\mathbb{R}^n$ (with the background metric chosen to be the Euclidean metric) is well known, see for example [65]. The stability theorems for Ricci flow on $\mathbb{R}^n$ obtained this way complement the existing results in the literature [84, 68, 72, 58]. One may also adapt the definition of the weighted spaces in this chapter to the (noncompact) real hyperbolic space $\mathbb{H}^n$ ($n \geq 3$) and prove interpolation properties accordingly. In this way one can obtain stability results that complement those in [61, 73].

Chapter 3 is organized as follows. In Section 3.2, we set up notations and recall the linearization formulae for the (curvature-normalized) Ricci flow. In Section 3.3, we study the spectrum of the elliptic operator in the linearized equation in all complex dimensions, and as a consequence, we obtain strict linear stability of the curvature-normalized Ricci-DeTurck flow at $(\mathbb{CH}^m, g_B)$ for $m \geq 2$. In Section 3.4, we define the unweighted space $\mathfrak{h}^{k+\alpha}$ and weighted space $\mathfrak{h}^{k+\alpha}_\tau$, and state an interpolation theorem (Theorem 3.4) for $\mathfrak{h}^{k+\alpha}_\tau$. The proof of Theorem 3.4 is technical, so we postpone it to Section 3.7. In Sections 3.5 and 3.6, we prove Theorems 3.1 and 3.2 respectively.

3.2 Preliminaries

Let (M^n, g) be an n-dimensional Riemannian manifold. The Riemann curvature tensor Rm of the metric g admits the decomposition

$$\mathrm{Rm} = \frac{R}{2n(n-1)} g \owedge g + \frac{1}{n-2} \overset{\circ}{\mathrm{Ric}} \owedge g + W,$$

where $\owedge$ stands for the Kulkarni-Nomizu product of symmetric tensors,

$$(P \owedge Q)_{ijkl} := P_{il}Q_{jk} + P_{jk}Q_{il} - P_{ik}Q_{jl} - P_{jl}Q_{ik},$$

$\overset{\circ}{\mathrm{Ric}} := \mathrm{Ric} - \frac{R}{n} g$ is the trace-free part of the Ricci tensor, and W is the Weyl curvature tensor.

If g is an Einstein metric, then $\overset{\circ}{\mathrm{Ric}}$ vanishes, and in this case the above decomposition simplies to

$$\mathrm{Rm} = \frac{R}{2n(n-1)} g \owedge g + W.$$

When $n = 2$ and 3, $W \equiv 0$, so the Einstein condition implies

$$\mathrm{Rm} = \frac{R}{2n(n-1)} g \owedge g,$$

which is equivalent to g having constant sectional curvature. Conversely, in all dimensions, constant sectional curvature implies that both $\overset{\circ}{\mathrm{Ric}}$ and W vanish. When $n \geq 4$, W vanishes if and only if (M, g) is locally conformally flat[2].

Let U be an open set of M^n. When possible, we take $U = M^n$, e.g., when $M^n = \mathbb{CH}^m$, then $U = \mathbb{CH}^m$. We denote by $\mathcal{T}^2$ the vector space of covariant two-tensor fields over U. We denote by $\mathcal{S}^2$ ($\mathcal{S}^2_c$, $\mathcal{S}^2_+$, respectively) the vector space of *symmetric* covariant two-tensor fields (with compact support, positive-definite, respectively) over U, and let $\mathcal{S}_2$ be the dual of $\mathcal{S}^2$. In the rest of the chapter, the regularity of the tensor fields will either be specified or dictated by the context. We let $\bigwedge_2$ denote the vector space of alternating two-vector fields over U. $\bigwedge_2$ has real dimension $\frac{n(n-1)}{2}$, and $\mathcal{S}^2$ has real dimension

[2]A Riemannian manifold (M, g) is locally conformally flat if around every point $p \in M$, there is a neighborhood U of p and a smooth function $f : U \to \mathbb{R}$ such that $(U, e^{2f} g)$ is Euclidean flat.

$\frac{n(n+1)}{2}$. Since $\bigwedge_2$ is isomorphic to the space of anti-symmetric covariant two-tensor fields over U, $\mathcal{T}^2 \cong \mathcal{S}^2 \oplus \bigwedge_2$. We define Ω^1 to be the space of one-forms over U.

We denote by $\mathcal{L}$ the Lie derivative, by $\delta = \delta_g : \mathcal{S}^2 \to \Omega^1$ the divergence operator (with respect to g), and by $\delta^* = \delta_g^* : \Omega^1 \to \mathcal{S}^2$ the formal L^2-adjoint of δ. $d\mu_g$ denotes the volume form of g, and we write $d\mu$ whenever there is no ambiguity. $\sharp : \Omega^1 \to \Gamma(TM)$ is the duality isomorphism induced by g.

We denote by $\langle \cdot, \cdot \rangle$ the tensor inner product with respect to g. Given an orthonormal frame field $\{e_i\}_{i=1}^n$ on U, we use the convention

$$\langle e_i \otimes e_j, e_k \otimes e_\ell \rangle = \delta_{ik}\delta_{j\ell}.$$

Define

$$e_i \wedge e_j := e_i \otimes e_j - e_j \otimes e_i, \quad e_i e_j := e_i \otimes e_j + e_j \otimes e_i,$$

then the set

$$\beta = \left\{ \frac{1}{\sqrt{2}} e_i \wedge e_j : 1 \leq i < j \leq n \right\}$$

forms an orthonormal frame field of $\bigwedge_2$ over U, and the set

$$\gamma = \left\{ \frac{1}{2} e_i e_i, \ \frac{1}{\sqrt{2}} e_i e_j : 1 \leq i < j \leq n \right\}$$

forms an orthonormal frame field of $\mathcal{S}_2$ over U.

The Riemann curvature tensor Rm induces by its symmetries an action $R_\bigwedge : \bigwedge_2 \to \bigwedge_2$ defined by

$$\langle R_\bigwedge (e_i \wedge e_j), e_k \wedge e_\ell \rangle = 4R(e_i, e_j, e_\ell, e_k).$$

Rm also induces an action $R_S : \mathcal{S}_2 \to \mathcal{S}_2$ defined by

$$\begin{aligned}\langle R_S(e_ie_j), e_pe_q\rangle &= R(e_i, e_p, e_q, e_j) + R(e_j, e_p, e_q, e_i)\\ &\quad + R(e_i, e_q, e_p, e_j) + R(e_j, e_q, e_p, e_i).\end{aligned}$$

Since $\mathcal{S}^2$ is dual to $\mathcal{S}_2$, R_S acts on $\mathcal{S}^2$ by the same action: if $h, k \in \mathcal{S}^2$, then

$$\langle R_S(h), k\rangle = 4R_{ipqj}h^{ij}k^{pq}.$$

We can represent $R_{\bigwedge}$ by a matrix R_β in the β-basis, and R_S by a matrix R_γ in the γ-basis. Both $R_{\bigwedge}$ and R_S encode the curvature information.

We now recall some well-known facts about the Ricci flow, see for example [41] or [27, Chapter 3]. The Ricci flow is a one-parameter family of Riemannian metrics $g(t)$, $0 \le t < T \le \infty$, evolving by

$$\frac{\partial}{\partial t}g = -2\operatorname{Ric}(g).$$

Because the Ricci curvature is invariant under the action of the inifinite group of diffeomorphisms of the manifold, Ricci flow is a weakly parabolic equation.

For $h \in \mathcal{S}^2$ sufficiently differentiable, the linearized Ricci flow is given by

$$\frac{\partial}{\partial t}h = \Delta_L h + \mathcal{L}_{(\delta G(h))^\sharp}g, \tag{3.1}$$

where $G(h) := h - \frac{1}{2}(\operatorname{Tr}_g h)g$, and Δ_L is the Lichnerowicz Laplacian. In local coordinates, we have

$$(\Delta_L h)_{ij} = (\Delta h)_{ij} + 2R_{ipqj}h^{pq} - R_i^k h_{kj} - R_j^k h_{ki}, \tag{3.2}$$

where $\Delta h = g^{ij}\nabla_i\nabla_j h$ is the rough Laplacian of h.

For $u \in \mathcal{S}^2_+$, define an operator $P_u(\cdot)$ on the metric g by

$$P_u(g) := -2\delta^*(\tilde{u}\delta(G(g,u))),$$

where $G(g,u) := u - \frac{1}{2}(\operatorname{Tr}_g u)g$, and $\tilde{u} : \Omega^1 \to \Omega^1$ given by $(\tilde{u}\beta)_j := g_{jk}u^{k\ell}\beta_\ell$. Then the Ricci-DeTurck flow is

$$\frac{\partial}{\partial t} g = -2\operatorname{Ric}(g) - P_u(g). \tag{3.3}$$

Equation (3.3) is strictly parabolic, and its linearization at g is

$$\frac{\partial}{\partial t} h = \Delta_L h - 2\lambda h.$$

Since we are interested in the dynamics of Ricci flow near g_B over the noncompact $\mathbb{CH}^m$ (or g_0 over the compact quotient M^n), we choose to study the curvature-normalized Ricci flow

$$\frac{\partial}{\partial t} g = -2(\operatorname{Ric}(g) + \lambda g), \tag{3.4}$$

so then g_B (or g_0) becomes a fixed point of this modified flow. g_B (or g_0) is also a fixed point of the curvature-normalized Ricci-DeTurck flow

$$\frac{\partial}{\partial t} g = -2(\operatorname{Ric}(g) + \lambda g) - P_{g_B}(g), \tag{3.5}$$

since it is straightforward to verify that $P_{g_B}(g_B) = 0$. The linearization of equation (3.5) at g_B is

$$\frac{\partial}{\partial t} h = \Delta_L h - 2\lambda h$$

by [41, Proposition 3.2].

To deduce dynamical stability from linear stability, we use the following version of Simonett's Stability Theorem. See [41, 55] for a more general version of the theorem, and [77] for the most general statement.

Theorem 3.3 (Simonett)**.** *Assume the following conditions hold:*

(B1) $\mathbb{X}_1 \overset{d}{\hookrightarrow} \mathbb{X}_0$ *and* $\mathbb{E}_1 \overset{d}{\hookrightarrow} \mathbb{E}_0$ *are continuous and dense inclusions of Banach spaces. For fixed* $0 < \beta < \alpha < 1$*,* $\mathbb{X}_\alpha$ *and* $\mathbb{X}_\beta$ *are continuous interpolation spaces corresponding to the inclusion* $\mathbb{X}_1 \overset{d}{\hookrightarrow} \mathbb{X}_0$*.*

(B2) Let

$$\frac{\partial}{\partial t} g = Q(g)g \tag{3.6}$$

be an autonomous quasilinar parabolic equation for $t \geq 0$*, with* $Q(\cdot) \in C^k(\mathbb{G}_\beta, \mathcal{L}(\mathbb{X}_1, \mathbb{X}_0))$ *for some positive integer* k *and some open set* $\mathbb{G}_\beta \subset \mathbb{X}_\beta$*.*

(B3) For each $\hat{g} \in \mathbb{G}_\beta$*, the domain* $D(Q(\hat{g}))$ *contains* $\mathbb{X}_1$*, and there exists an extension* $\hat{Q}(\hat{g})$ *of* $Q(\hat{g})$ *to a domain* $D(\hat{Q}(\hat{g}))$ *containing* $\mathbb{E}_1$*.*

(B4) For each $\hat{g} \in \mathbb{G}_\alpha := \mathbb{G}_\beta \cap \mathbb{X}_\alpha$*,* $\hat{Q}(\hat{g}) \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_0)$ *generates a strongly continuous analytic semigroup on* $\mathcal{L}(\mathbb{E}_0, \mathbb{E}_0)$*.*

(B5) For each $\hat{g} \in \mathbb{G}_\alpha$*,* $Q(\hat{g})$ *agrees with the restriction of* $\hat{Q}(\hat{g})$ *to the dense subset* $D(Q(\hat{g})) \subset \mathbb{X}_0$*.*

(B6) *Let* $(\mathbb{E}_0, D(\hat{Q}(\cdot)))_\theta$ *be the continuous interpolation space. Define the set* $(\mathbb{E}_0, D(\hat{Q}(\cdot)))_{1+\theta} := \{x \in D(\hat{Q}(\cdot)) : D(\hat{Q}(\cdot))(x) \in (\mathbb{E}_0, D(\hat{Q}(\cdot))_\theta\}$ *endowed with the graph norm of* $\hat{Q}(\cdot)$ *with respect to* $(\mathbb{E}_0, D(\hat{Q}(\cdot)))_\theta$*. Then* $\mathbb{X}_0 \cong (\mathbb{E}_0, D(\hat{Q}(\cdot)))_\theta$ *and* $\mathbb{X}_1 \cong (\mathbb{E}_0, D(\hat{Q}(\cdot)))_{1+\theta}$ *for some* $\theta \in (0,1)$*.*

(B7) $\mathbb{E}_1 \overset{d}{\hookrightarrow} \mathbb{X}_\beta \overset{d}{\hookrightarrow} \mathbb{E}_0$ *with the property that there are constants* $C > 0$ *and* $\theta \in (0,1)$ *such that for all* $x \in \mathbb{E}_1$*, one has*

$$\|x\|_{\mathbb{X}_\beta} \leq C\|x\|_{\mathbb{E}_0}^{1-\theta}\|x\|_{\mathbb{E}_1}^{\theta}.$$

For each $\alpha \in (0,1)$*, let* $g_0 \in \mathbb{G}_\alpha$ *be a fixed point of equation* (3.6)*. Suppose that the spectrum of the linearized operator* $DQ|_{g_0}$ *is contained in the set* $\{z \in \mathbb{C} : \Re(z) \leq -\varepsilon\}$ *for some constant* $\varepsilon > 0$*. Then there exist constants* $\omega \in (0, \varepsilon)$ *and* $d_0, C_\alpha > 0$*,* C_α *independent of* g_0*, such that for each* $d \in (0, d_0]$*, one has*

$$\|\tilde{g}(t) - g_0\|_{\mathbb{X}_1} \leq \frac{C_\alpha}{t^{1-\alpha}} e^{-\omega t}\|\tilde{g}(0) - g_0\|_{\mathbb{X}_\alpha}$$

for all solutions $\tilde{g}(t)$ *of equation* (3.6) *with* $\tilde{g}(0) \in B(\mathbb{X}_\alpha, g_0, d)$*, the open ball of radius* d *centered at* g_0 *in the space* $\mathbb{X}_\alpha$*, and for all* $t \geq 0$*.*

3.3 Linearization and linear stability

3.3.1 Linearization in any complex dimension

Let $m \in \mathbb{N}$ and consider $(\mathbb{CH}^m, g_B)$. Recall that g_B is Einstein, $\mathrm{Ric}(g_B) = -\lambda g_B$ $(\lambda > 0)$, and g_B has constant holomorphic sectional curvature $-c$

$(c > 0)$. In particular, $(\mathbb{CH}^m, g_B)$ is a Riemannian manifold of real dimension $n = 2m$ with quarter-pinched sectional curvature $K \in [-c, -c/4]$. The results in this section remain valid for any smooth closed quotient (M^n, g_0) of $(\mathbb{CH}^m, g_B)$.

Recall that for $h \in \mathcal{S}^2$, the linearization of equation (3.5) at g_B is

$$\frac{\partial}{\partial t} h = \Delta_L h - 2\lambda h. \tag{3.7}$$

Define the linear operator $A : \mathcal{S}^2 \to \mathcal{S}^2$ by

$$Ah := \Delta_L h - 2\lambda h.$$

Equation (3.2) implies that A is the rough Laplacian plus zero-order terms, so A is a strictly elliptic and self-adjoint operator on $L^2(\mathcal{S}^2_c)$. Since g_B is Einstein, the Ricci tensor (after raising one index) acts on $(1,1)$-tensors with eigenvalues $-\lambda$. So we have

$$(Ah)_{ij} = (\Delta h)_{ij} + 2R_{ipqj} h^{pq}.$$

For now we denote by (M, g) an arbitrary complete Einstein manifold. M is either noncompact, e.g., $(\mathbb{CH}^m, g_B)$, or closed, e.g., (M^n, g_0). We let $(\cdot, \cdot)$ be the L^2-pairing on S^2_c defined by

$$(h, k) = \int_M \langle h, k \rangle d\mu.$$

Remark 3.1. *Recall our convention that $\langle e_i \otimes e_j, e_k \otimes e_\ell \rangle = \delta_{ik}\delta_{j\ell}$, then*

$$(h, k) = \int_M \langle h^{ij} e_i e_j, k^{pq} e_p e_q \rangle d\mu = 4 \int_M h^{ij} k_{ij} d\mu.$$

For $h \in \mathcal{S}^2_c$ that is sufficiently differentiable, we have

$$
\begin{aligned}
(Ah, h) &= \int_M \langle \Delta h, h \rangle \, d\mu + 8 \int_M R_{ipqj} h^{pq} h^{ij} d\mu \\
&= \int_M \langle \Delta h, h \rangle \, d\mu + 2 \int_M \langle R_S(h), h \rangle d\mu.
\end{aligned}
$$

Then we integrate by parts to get

$$
(Ah, h) = -\int_M \langle \nabla h, \nabla h \rangle \, d\mu + 2 \int_M \langle R_S(h), h \rangle d\mu. \tag{3.8}
$$

We let $|h|^2 := \langle h, h \rangle$, $\|h\|^2 := (h, h)$; for a function f, $\|f\|^2$ is its L^2-norm.

Lemma 3.2. *On a negatively curved Einstein manifold (M, g) with* $\mathrm{Ric}(g) = -\lambda g$ *($\lambda > 0$), given $h \in \mathcal{S}^2_c$, define a covariant three-tensor by*

$$
T_{ijk} := \nabla_k h_{ij} - \nabla_i h_{jk}.
$$

Then

$$
\|\nabla h\|^2 = \frac{1}{2}\|T\|^2 + \|\delta h\|^2 + \lambda \|h\|^2 + \int_M \langle R_S(h), h \rangle d\mu.
$$

This is Koiso's Bochner formula [59], and we include its proof here for completeness.

Proof. $T_{ijk} := \nabla_k h_{ij} - \nabla_i h_{jk}$, then

$$
\begin{aligned}
\|T\|^2 &= \int_M 4 \left\langle \nabla_k h_{ij} - \nabla_i h_{jk}, \nabla^k h^{ij} - \nabla^i h^{jk} \right\rangle d\mu \\
&= 2\|\nabla h\|^2 - 8 \int_M \nabla_k h_{ij} \nabla^i h^{jk} d\mu.
\end{aligned}
$$

We integrate by parts and commute the covariant derivatives to obtain

$$\begin{aligned}
-4\int_M \nabla_k h_{ij}\nabla^i h^{jk} d\mu &= 4\int_M h^i_j \nabla_k\nabla_i h^{jk} d\mu \\
&= 4\int_M h^i_j(\nabla_i\nabla_k h^{jk} + R^j_{kip}h^{pk} + R^k_{kip}h^{jp}) d\mu \\
&= -\|\delta h\|^2 + 4\int_M h^i_j R_{ip}h^{jp} d\mu + 4\int_M h^i_j R^j_{kip}h^{pk} d\mu \\
&= -\|\delta h\|^2 - \lambda\|h\|^2 - \int_M \langle R_S(h), h\rangle d\mu.
\end{aligned}$$

Rearranging the terms proves the lemma. □

Therefore, equation (3.8) becomes

$$(Ah, h) = -\frac{1}{2}\|T\|^2 - \|\delta h\|^2 - \lambda\|h\|^2 + \int_M \langle R_S(h), h\rangle d\mu. \tag{3.9}$$

Remark 3.3. *Applying equation* (3.9) *on a closed Riemannian manifold of dimension* $n \geq 3$ *and constant sectional curvature* $K = -1$ *(after normalizing the metric), then*

$$\int_M \langle R_S(h), h\rangle d\mu = 4\int_M -(g_{ij}g_{pq} - g_{iq}g_{pj})h^{pq}h^{ij} d\mu = -\|\operatorname{Tr}_g h\|^2 + \|h\|^2.$$

Since in this case $\lambda = n-1$, *then for* $h \not\equiv 0$, *equation* (3.9) *implies*

$$(Ah, h) \leq -(n-2)\|h\|^2.$$

Thus, the curvature-normalized Ricci flow is strictly linearly stable at a closed negatively curved space form. This was proved in [56, Appendix A].

Remark 3.4. *In Remark 3.3, the dimension assumption* $n \geq 3$ *is crucial for linear stability. On a real two-dimensional (complex one-dimensional) surface* *of genus $\gamma > 1$, however, the operator A is* not *strictly linearly stable. A has a nullspace of complex dimension $3\gamma - 3$, which is isomorphic to the space of holomorphic quadratic differentials, hence with the cotangent space to the Teichmüller space.*

3.3.2 Strict linear stability in complex dimension two or higher

Let $m \in \mathbb{N}$. Let U be the single geodesic normal coordinate chart covering $\mathbb{CH}^m$ (or one of a finite atlas of coordinate charts covering M^n). We fix an orthonormal frame field $\{e_i, e_{i+1} : i = 2k-1, k = 1, 2, \ldots, m\}$ over U such that the complex structure J acts on this frame field by

$$J : \{e_i, e_{i+1}\} \mapsto \{e_{i+1}, -e_i\}, \text{ for each } i = 2k-1,\ k = 1, 2, \ldots, m.$$

We abuse the notation and define the action J on the indices by

$$J(s) = t, \text{ if } J(e_s) = e_t \text{ or } J(e_s) = -e_t.$$

For example, $J(1) = 2,\ J(2) = 1$.

We now define a canonical frame field γ for $\mathcal{S}_2$ of real dimension $m(2m+1)$ in three groups, denoted by γ_I, γ_{II}, and γ_{III}, respectively. We define

$$\begin{aligned} \gamma_I^i &:= \frac{1}{2} e_i e_i, \quad i = 1, 2, \ldots, 2m; \\ \gamma_{II}^j &:= \frac{1}{\sqrt{2}} e_{2j-1} e_{2j}, \quad j = 1, 2, \ldots, m. \end{aligned}$$

We define $2m(m-1)$ basis elements of γ_{III} by

$$\left\{ \begin{array}{lll} \gamma_{III}^p & := & \frac{1}{\sqrt{2}} e_s e_t, \\ \gamma_{III}^{p+1} & := & \frac{1}{\sqrt{2}} e_{J(s)} e_{J(t)}, \end{array} \right. \quad \text{if } 1 \leq s < t \leq m, \text{ and } J(s) \neq t,$$

which for convenience we index by $p = 1, 3, 5, \ldots, 2m(m-1)-1$. For example, $\gamma_{III}^1 = \frac{1}{\sqrt{2}}e_1e_3, \gamma_{III}^2 = \frac{1}{\sqrt{2}}e_2e_4$; $\gamma_{III}^3 = \frac{1}{\sqrt{2}}e_1e_4, \gamma_{III}^4 = \frac{1}{\sqrt{2}}e_2e_3$; $\gamma_{III}^5 = \frac{1}{\sqrt{2}}e_1e_5, \gamma_{III}^6 = \frac{1}{\sqrt{2}}e_2e_6$, etc.

We define three matrices A_m, B_m, and C_m. A_m is a $2m \times 2m$ matrix given by

$$A_m = \begin{pmatrix} D & E & \cdots & \cdots & E \\ E & D & E & \cdots & \vdots \\ \vdots & E & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & E \\ E & \cdots & \cdots & E & D \end{pmatrix},$$

where

$$D = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

B_m is the $m \times m$ diagonal matrix whose diagonal entries are -4. C_m is a $2m(m-1) \times 2m(m-1)$ matrix and it is block diagonal given by

$$C_m = \begin{pmatrix} F & & & \\ & F & & \\ & & \ddots & \\ & & & F \end{pmatrix},$$

where

$$F = \begin{pmatrix} -1 & 3 & & \\ 3 & -1 & & \\ & & -1 & -3 \\ & & -3 & -1 \end{pmatrix}.$$

Then we have the following.

Proposition 3.5. *For each $m \in \mathbb{N}$, R_S is represented in the canonical γ-basis by the block diagonal matrix R_γ with*

$$R_\gamma = -\frac{c}{4}\begin{pmatrix} A_m & & \\ & B_m & \\ & & C_m \end{pmatrix}.$$

We have the advantage of knowing the full information on the Riemann curvature tensor of g_B.

Lemma 3.6. *The non-zero components of the Riemann curvature tensor of g_B are given by the following:*

$$\text{if } J(i) = j, \text{ then } R(e_i, e_j, e_j, e_i) = -c;$$

$$\text{if } J(i) \neq j, \text{ then } R(e_i, e_j, e_j, e_i) = -c/4;$$

$$\text{if } k < \ell, p < q, J(k) = \ell, J(p) = q, \text{ then } \left\{ \begin{array}{rcr} R(e_k, e_\ell, e_q, e_p) & = & -c/2, \\ R(e_k, e_p, e_q, e_\ell) & = & -c/4, \\ R(e_k, e_q, e_p, e_\ell) & = & c/4. \end{array} \right.$$

Proof. On a Kähler manifold (M, g) of constant holomorphic sectional curvature $-c$ ($c > 0$), e.g., $(\mathbb{CH}^m, g_B)$ or (M^n, g_0), if X, Y are orthonormal vectors in T_pM, then the Riemannian sectional curvature $K(X, Y)$ at p is given by O'Neill's formula ([57])

$$K(X, Y) = -\frac{c}{4}\left(1 + 3g(JX, Y)^2\right).$$

For $X, Y, Z, W \in T_pM$, we also have

$$\begin{aligned} R(X, Y, W, Z) &= R(X, Y, JW, JZ) = R(JX, JY, W, Z) \\ &= R(JX, JY, JW, JZ). \end{aligned}$$

We compute

$$
\begin{aligned}
R(e_1, e_2, e_2, e_1) = K(e_1, e_2) &= -\frac{c}{4}\left(1 + 3g(Je_1, e_2)^2\right) \\
&= -\frac{c}{4}\left(1 + 3g(e_2, e_2)^2\right) = -c,
\end{aligned}
$$

$$
\begin{aligned}
R(e_1, e_3, e_3, e_1) = K(e_1, e_3) &= -\frac{c}{4}\left(1 + 3g(Je_1, e_3)^2\right) \\
&= -\frac{c}{4}\left(1 + 3g(e_2, e_3)^2\right) = -\frac{c}{4},
\end{aligned}
$$

$$
\begin{aligned}
R(e_1, e_3, e_4, e_2) &= R(e_1, e_3, Je_4, Je_2) \\
&= R(e_1, e_3, -e_3, -e_1) = -\frac{c}{4},
\end{aligned}
$$

$$
\begin{aligned}
R(e_1, e_4, e_3, e_2) &= R(e_1, e_4, Je_3, Je_2) \\
&= R(e_1, e_4, e_4, -e_1) = \frac{c}{4}.
\end{aligned}
$$

Using the first Bianchi identity[3] we compute

$$
\begin{aligned}
R(e_1, e_2, e_4, e_3) &= -R(e_1, e_4, e_3, e_2) - R(e_1, e_3, e_2, e_4) \\
&= -R(e_1, e_4, Je_3, Je_2) - R(e_1, e_3, Je_2, Je_4) \\
&= -R(e_1, e_4, e_4, -e_1) - R(e_1, e_3, -e_1, -e_3) \\
&= R(e_1, e_4, e_4, e_1) + R(e_1, e_3, e_3, e_1) \\
&= -\frac{c}{2},
\end{aligned}
$$

[3]The first Bianchi identity says that $R(X, Y, Z, W) + R(X, Z, W, Y) + R(X, W, Y, Z) = 0$ for any $X, Y, Z, W \in T_pM$.

Noting the patterns in the above computation, we then quickly obtain the other non-zero components. For example, $R(e_5, e_6, e_6, e_5) = -c$, $R(e_1, e_5, e_5, e_1) = R(e_1, e_5, e_6, e_2) = -R(e_1, e_6, e_5, e_2) = -c/4$, $R(e_1, e_2, e_6, e_5) = -c/2$, etc.

All the remaining components are zero, for example,

$$\begin{aligned}
R(e_1, e_2 + e_3, e_2 + e_3, e_1) &= R(e_1, e_2, e_2, e_1) + R(e_1, e_3, e_3, e_1) \\
&\quad + R(e_1, e_2, e_3, e_1) + R(e_1, e_3, e_2, e_1) \\
&= -\frac{5c}{4} + 2R(e_1, e_2, e_3, e_1),
\end{aligned}$$

$$\begin{aligned}
R(e_1, e_2 + e_3, e_2 + e_3, e_1) &= 2R\left(e_1, \frac{e_2 + e_3}{\sqrt{2}}, \frac{e_2 + e_3}{\sqrt{2}}, e_1\right) \\
&= -2\frac{c}{4}\left(1 + 3g\left(J\left(\frac{e_2 + e_3}{\sqrt{2}}\right), e_1\right)^2\right) \\
&= -2\frac{c}{4}\left(1 + 3g\left(\frac{-e_1 + e_4}{\sqrt{2}}, e_1\right)^2\right) \\
&= -\frac{c}{2}(1 + \frac{3}{2}) = -\frac{5c}{4}.
\end{aligned}$$

So $R(e_1, e_2, e_3, e_1) = 0$. $\square$

Proposition 3.5 is now a direct consequence of Lemma 3.6.

Proof of Proposition 3.5. In the γ-basis defined in Section 3.3.2 and using

Lemma 3.6, for example,

$$\begin{aligned}
\left\langle R_\gamma\left(\frac{e_1e_1}{2}\right), \frac{e_1e_1}{2}\right\rangle &= R(e_1,e_1,e_1,e_1) = 0,\\
\left\langle R_\gamma\left(\frac{e_1e_1}{2}\right), \frac{e_2e_2}{2}\right\rangle &= R(e_1,e_2,e_2,e_1) = -c,\\
\left\langle R_\gamma\left(\frac{e_1e_1}{2}\right), \frac{e_3e_3}{2}\right\rangle &= R(e_1,e_3,e_3,e_1) = -\frac{c}{4},
\end{aligned}$$

$$\begin{aligned}
\left\langle R_\gamma\left(\frac{e_1e_2}{\sqrt{2}}\right), \frac{e_1e_2}{\sqrt{2}}\right\rangle &= R(e_1,e_2,e_1,e_2) = c,\\
\left\langle R_\gamma\left(\frac{e_1e_2}{\sqrt{2}}\right), \frac{e_3e_4}{\sqrt{2}}\right\rangle &= R(e_1,e_3,e_4,e_2) + R(e_1,e_4,e_3,e_2) = 0,\\
\left\langle R_\gamma\left(\frac{e_1e_3}{\sqrt{2}}\right), \frac{e_2e_4}{\sqrt{2}}\right\rangle &= R(e_1,e_2,e_4,e_3) + R(e_1,e_4,e_2,e_3) = -\frac{3c}{4},\\
\left\langle R_\gamma\left(\frac{e_1e_4}{\sqrt{2}}\right), \frac{e_2e_3}{\sqrt{2}}\right\rangle &= R(e_1,e_2,e_3,e_4) + R(e_1,e_3,e_2,e_4) = \frac{3c}{4}.
\end{aligned}$$

The remaining entries of R_γ are computed analogously. □

Lemma 3.7. *For each $m \in \mathbb{N}$, the largest eigenvalue of R_γ is c.*

Proof. R_γ is a block diagonal matrix, so its eigenvalues are $-\frac{c}{4}$ times the eigenvalues of A_m, B_m, and C_m [48]. The eigenvalues of B_m are -4. The matrix C_m is block diagonal, so its eigenvalues are given by the eigenvalues of F, which are $\{2, 2, -4, -4\}$. It remains to understand the eigenvalues of A_m.

We write vectors in $\mathbb{R}^{2m}$ as column vectors. Define $X \in \mathbb{R}^{2m}$ by

$$X^T = (\underbrace{1, 1, \cdots, 1, 1}_{2m \text{ of them}}).$$

Define $Y_i \in \mathbb{R}^{2m}$, $i = 1, 2, \ldots, m-1$, by

$$Y_i^T = (-1, -1, \underbrace{0, \cdots, 0}_{\text{all } 0}, \underbrace{1}_{(2i+1)\text{-th entry}}, 1, \underbrace{0, \cdots, 0}_{\text{all } 0}).$$

Define $Z_i \in \mathbb{R}^{2m}$, $i = 1, 2, \ldots, m$, by

$$Z_i^T = (\underbrace{0, \cdots, 0}_{\text{all } 0}, \underbrace{-1}_{(2i-1)\text{-th entry}}, 1, \underbrace{0, \cdots, 0}_{\text{all } 0}).$$

Then by a direct computation, we have

$$A_m X = 2(m+1)X;$$

$$A_m Y_i = 2Y_i, \quad i = 1, 2, \ldots, m-1;$$

$$A_m Z_i = -4Z_i, \quad i = 1, 2, \ldots, m.$$

Thus, the eigenvalues of A_m are

$$\{2(m+1), \underbrace{2, \cdots, 2}_{(m-1) \text{ of them}}, \underbrace{-4, \cdots, -4}_{m \text{ of them}}\}.$$

The lemma follows after multiplying the eigenvalues of A_m, B_m, and C_m by $-\frac{c}{4}$. $\square$

Therefore, we have the following estimate.

Proposition 3.8. *Let $m \in \mathbb{N}$. For all $h \in \mathcal{S}_c^2 \setminus \{0\}$,*

$$(Ah, h) \leq -\frac{m-1}{2} c \|h\|^2, \quad c > 0. \tag{3.10}$$

In particular, the curvature-normalized Ricci-DeTurck flow (3.5) *is strictly linearly stable at* $(\mathbb{CH}^m, g_B)$ *or* (M^n, g_0) *when* $m \geq 2$.

Proof. The sum of the entries of $-\frac{c}{4}A_m$ is the scalar curvature R of g_B, so $R = -m(m+1)c$, and hence

$$\lambda = -\frac{R}{2m} = \frac{m+1}{2}c.$$

Let λ_S denote the largest eigenvalue of R_γ, then $\lambda_S = c$ by Lemma 3.7. Recall the variational characterization

$$\lambda_S = \sup_{h \in \mathcal{S}^2_c \setminus \{0\}} \frac{\langle R_S h, h \rangle}{\langle h, h \rangle}.$$

Then equation (3.9) implies

$$\begin{aligned}(Ah, h) &\leq -\lambda \|h\|^2 + \int \langle R_S(h), h \rangle \\ &\leq -\frac{m+1}{2} c\|h\|^2 + c\|h\|^2 \\ &\leq -\frac{m-1}{2} c\|h\|^2, \quad c > 0.\end{aligned}$$

The asserted strict linear stability follows if $m \geq 2$. $\square$

Remark 3.9. *On a closed manifold (M^n, g_0), we can assume $h \in \mathcal{S}^2$. On the complete noncompact $(\mathbb{CH}^m, g_B)$, we will relax the assumption $h \in \mathcal{S}^2_c$, cf. Corollary 3.14.*

Inequality (3.10) is sharp since when $m = 1$, the operator A has a nontrivial nullspace, cf. Remark 3.4. On the other hand, if on M^n there is a smooth one-parameter family of complex hyperbolic metrics with some fixed complex structure, then A would have a nontrivial null eigenspace. Proposition 3.8 shows that the operator A has trivial null eigenspace when $m \geq 2$. Hence, we recover the following well-known result: the moduli space of complex hyperbolic metrics on a closed 4-manifold is locally rigid in the sense that if a complex hyperbolic metric g is near g_0 in some norm, then g differs from g_0 by a homothetic scaling and a diffeomorphism on M^4 [50, 52, 60]. Moreover, we have extended this local rigidity to any closed $2m$-manifold when $m \geq 2$.

3.4 (Weighted) little Hölder spaces

On a closed Riemannian manifold M^n admitting a complex hyperbolic metric, we fix a background metric and a finite atlas $\{U_\upsilon\}_{1\leq\upsilon\leq\Upsilon}$ of coordinate charts covering M^n. Given a smooth $h \in \mathcal{S}^2$ over M^n, for each integer $k \geq 0$ and real number $\alpha \in (0,1)$, we denote

$$[h_{ij}]_{k+\alpha;U_\upsilon} := \sup_{|\ell|=k} \sup_{x\neq y\in U_\upsilon} \frac{\left|\nabla^\ell h_{ij}(x) - \nabla^\ell h_{ij}(y)\right|}{d(x,y)^\alpha},$$

where ℓ is a multi-index and $\nabla^\ell h_{ij} = \nabla_1^{\ell_1}\nabla_2^{\ell_2}\cdots\nabla_n^{\ell_n} h_{ij}$.

We define the $(k+\alpha)$-Hölder norm of h by

$$\|h\|_{k+\alpha} := \sup_{\substack{1\leq i,j\leq n,\\ 1\leq\upsilon\leq\Upsilon}} \left(\sum_{m=0}^{k} \sup_{|\ell|=m} \sup_{x\in U_\upsilon} \left|\nabla^\ell h_{ij}(x)\right| + [h_{ij}]_{k+\alpha;U_\upsilon}\right).$$

The components $\{h_{ij}\}$ are with respect to the fixed atlas $\{U_\upsilon\}_{1\leq\upsilon\leq\Upsilon}$, whereas ∇, $|\cdot|$, and the distance function d are computed with respect to a fixed background metric. Different finite atlases or background metrics yield equivalent Hölder norms.

The little Hölder space $\mathfrak{h}^{k+\alpha}$ on the closed manifold M^n is defined to be the completion of C^∞ symmetric covariant two-tensor fields in $\|\cdot\|_{k+\alpha}$. The space $\mathfrak{h}^{k+\alpha}$ is not separable [41]. In particular, $h \in \mathfrak{h}^{k+\alpha}$ has the following property [64, Proposition 0.2.1]:

$$\lim_{t\to 0+} \sup_{|\ell|=k} \sup_{\substack{x\neq y\in U_\upsilon,\\ d(x,y)\leq t}} \frac{\left|\nabla^\ell h_{ij}(x) - \nabla^\ell h_{ij}(y)\right|}{d(x,y)^\alpha} = 0$$

for all $1 \leq \upsilon \leq \Upsilon$ and $1 \leq i,j \leq n$. This property also defines $\mathfrak{h}^{k+\alpha}$ [64].

On a complete noncompact manifold the definition of the Hölder norm depends on the choice of the atlas and the background metric. On $\mathbb{CH}^m$, $m \in \mathbb{N}$, we will use the single geodesic normal coordinate chart given by the exponential map at the origin and choose the background metric to be g_B. In this case, $|\cdot|$, $\|\cdot\|$, and d are computed with respect to g_B.

Let $h \in \mathcal{S}^2$, h is not necessarily compactly supported over $\mathbb{CH}^m$. To apply equation (3.9) and Proposition 3.8, h should decay at spatial infinity with the decay rate dictated by the geometry under consideration. To apply Simonett's Stability Theorem to deduce dynamical stability from linear stability, h should also belong to a Banach space that satisfies certain interpolation properties. These considerations lead us to define the following suitably weighted little-Hölder spaces.

We first set up some notations. Let B_R be a geodesic ball of radius R centered at the origin of our single chart. Fix $m \in \mathbb{N}$, consider the open covering of $\mathbb{CH}^m$ by the family $\{A_N\}_{N\in\mathbb{N}}$ of overlapping annuli and a disk defined by $A_1 := B_4$, and $A_N := B_{N+3} \setminus B_{N-1}$ for $N \geq 2$. If $x, y \in A_N$, denote $d_x^N := d(x, \partial A_N)$, $d_{x,y}^N := \min\{d_x, d_y\}$, and we write $d_x, d_{x,y}$ whenever there is no ambiguity. Given a smooth $h \in \mathcal{S}^2$ over $\mathbb{CH}^m$, for integers $k, q \geq 0$, multi-index ℓ, and $\alpha \in (0,1)$, we let

$$|h_{ij}|'_{q;A_N} := \sup_{|\ell|=q} \sup_{x\in A_N} d_x^q |\partial^\ell h_{ij}(x)|,$$

$$[h_{ij}]'_{k+\alpha;A_N} := \sup_{|\ell|=k} \sup_{x\neq y\in A_N} d_{x,y}^{k+\alpha} \frac{|\partial^\ell h_{ij}(x) - \partial^\ell h_{ij}(y)|}{d(x,y)^\alpha}.$$

For each $m \in \mathbb{N}$, we fix $\tau = (m+\xi)/2$ where $\xi > 0$ and define the τ-weighted $(k+\alpha)$-Hölder norm of h by

$$\|h\|_{k+\alpha;\tau} := \sup_{1\le i,j\le n} \sup_{N\in\mathbb{N}} e^{N\tau} \left(\sum_{q=0}^{k} |h_{ij}|'_{q;A_N} + [h_{ij}]'_{k+\alpha;A_N} \right).$$

The τ-weighted little Hölder space $\mathfrak{h}_\tau^{k+\alpha}$ on the complete noncompact manifold $\mathbb{CH}^m$ is defined to be the closure of C_c^∞ symmetric covariant two-tensor fields in $\|\cdot\|_{k+\alpha;\tau}$.

Similar to the little Hölder space $\mathfrak{h}^{k+\alpha}$, we have the following infinitesimal property for $h \in \mathfrak{h}_\tau^{k+\alpha}$.

Lemma 3.10. *If $h \in \mathfrak{h}_\tau^{k+\alpha}$, define*

$$F_h(t) := \sup_{1\le i,j\le n} \sup_{N\in\mathbb{N}} e^{N\tau} \left(\sup_{|\ell|=k} \sup_{\substack{x\neq y\in A_N,\\ d(x,y)\le t}} d_{x,y}^{k+\alpha} \frac{|\partial^\ell h_{ij}(x) - \partial^\ell h_{ij}(y)|}{d(x,y)^\alpha} \right),$$

then

$$\lim_{t\to 0+} F_h(t) = 0. \tag{3.11}$$

Proof. Given $h \in \mathfrak{h}_\tau^{k+\alpha}$, there exists a sequence of C_c^∞ symmetric two-tensor fields $\{h_n\}_{n\in\mathbb{N}}$ that converge to h in $\mathfrak{h}_\tau^{k+\alpha}$. Note that each h_n satisfies property 3.11. Then we estimate

$$F_h(t) \le F_{(h-h_n)}(t) + F_{h_n}(t).$$

In particular,

$$F_{(h-h_n)}(t) \le \|h-h_n\|_{k+\alpha;\tau}.$$

For $\varepsilon > 0$, choose n large such that $\|h - h_n\|_{k+\alpha;\tau} < \varepsilon/2$, and choose $\delta > 0$ such that $F_{h_n}(t) < \varepsilon/2$ if $t < \delta$, then $F_h(t) < \varepsilon$ if $t < \delta$. $\square$

Remark 3.11. *From now on, we shorten some expressions. For integer $k \geq 0$, ∇^k (or ∂^k) means first applying ∇^ℓ (or ∂^ℓ) and then taking the supremum over $|\ell| = k$. We omit the indices of h, and the result will follow after either summing or taking the supremum over $1 \leq i, j \leq n = 2m$.*

We denote by $W^2_1(\mathbb{CH}^m)$ the Sobolev space of symmetric two-tensor fields h over $\mathbb{CH}^m$ such that

$$\sum_{i,j=1}^{2m} \int_{\mathbb{CH}^m} \left(|h_{ij}|^2 + |\nabla h_{ij}|^2\right) d\mu_{g_B} < \infty,$$

where ∇ is computed with respect to the background metric g_B. We denote by $\hookrightarrow$ a continuous embedding.

Lemma 3.12. *Let $k, m \in \mathbb{N}$, $\alpha \in (0,1)$, and fix $\tau > m/2$. Then $\mathfrak{h}^{k+\alpha}_\tau \hookrightarrow W^2_1(\mathbb{CH}^m)$.*

Proof. Recall that B_R denotes a geodesic ball of radius R centered at the origin in $\mathbb{CH}^m$. The volume of B_R, denoted by $V(B_R)$, has exponential growth rate m, i.e., there is a positive constant $\tilde{C} = \tilde{C}(m)$ such that $V(B_R) \leq \tilde{C}e^{mR}$, see for example [39]. We know $\nabla h = \partial h + \Gamma * h$, where $*$ denotes contraction using g_B, and Γ is uniformly bounded by some constant C in geodesic normal coordinates on the homogeneous space $(\mathbb{CH}^m, g_B)$. In what follows and the rest of the chapter, "$A \lesssim B$" means $A \leq CB$, where $C > 0$ is a constant that may change from line to line.

Let $h \in \mathfrak{h}^{k+\alpha}_{\tau}$. Recall that $\tau = (m+\xi)/2$ where $\xi > 0$. If $x \in B_2$, then $x \in A_1$ with $d_x \geq 1$, so we have

$$\begin{aligned}
\int_{B_2} \left(|h(x)|^2 + |\nabla h(x)|^2\right) d\mu_{g_B}(x) &\lesssim \int_{B_2} \left(|h(x)|^2 + |\partial h(x)|^2\right) d\mu_{g_B}(x) \\
&\leq \int_{B_2} \left[(|h(x)|'_{0;A_1})^2 + d_x^{-2}(|h(x)|'_{1;A_1})^2\right] d\mu_{g_B}(x) \\
&\leq \int_{B_2} \left[(|h(x)|'_{0;A_1})^2 + (|h(x)|'_{1;A_1})^2\right] d\mu_{g_B}(x) \\
&\leq \|h\|^2_{k+\alpha;\tau} e^{-2\tau} \int_{B_2} d\mu_{g_B} \\
&\lesssim \|h\|^2_{k+\alpha;\tau} e^{-2\tau} e^{2m} \\
&= \|h\|^2_{k+\alpha;\tau} e^{-\xi} e^{-m} e^{2m} \\
&= e^{m-\xi} \|h\|^2_{k+\alpha;\tau}.
\end{aligned}$$

Similarly, if $x \in B_{2N} \setminus B_{2N-2}$, $N \geq 2$, then $x \in A_{2N-2} = B_{2N+1} \setminus B_{2N-3}$ with $d_x \geq 1$. So $|h(x)| \leq |h|'_{0;A_{2N-2}}$, $|\partial h(x)| \leq d_x^{-1} |h|'_{1;A_{2N-2}} \leq |h|'_{1;A_{2N-2}}$, and we estimate

$$\begin{aligned}
\int_{B_{2N} \setminus B_{2N-2}} \left(|h(x)|^2 + |\nabla h(x)|^2\right) d\mu_{g_B}(x) &\lesssim e^{-2(2N-2)\tau} \|h\|^2_{k+\alpha;\tau} \int_{B_{2N} \setminus B_{2N-2}} d\mu_{g_B}(x) \\
&\lesssim e^{-(2N-2)\xi} \|h\|^2_{k+\alpha;\tau} e^{-(2N-2)m} e^{2Nm} \\
&= e^{2m-(2N-2)\xi} \|h\|^2_{k+\alpha;\tau}.
\end{aligned}$$

Then, with $B_0 = \emptyset$, we have

$$\begin{aligned}\int_{B_{2N}} \left(|h(x)|^2 + |\nabla h(x)|^2\right) d\mu_{g_B}(x) &= \sum_{K=1}^{N} \int_{B_{2K} \setminus B_{2K-2}} \left(|h(x)|^2 + |\nabla h(x)|^2\right) d\mu_{g_B}(x) \\ &\lesssim \|h\|^2_{k+\alpha;\tau} \left(e^{m-\xi} + \sum_{K=2}^{N} e^{2m-(2K-2)\xi} \right) \\ &\leq \|h\|^2_{k+\alpha;\tau} \left(e^{m-\xi} + e^{2m} \frac{e^{-2\xi}}{1 - e^{-2\xi}} \right).\end{aligned}$$

Let $N \to \infty$, we see $h \in W^2_1(\mathbb{CH}^m)$. $\square$

Lemma 3.13. *Let $k, m \in \mathbb{N}$, $\alpha \in (0,1)$, and fix $\tau > m/2$. Then for all $h \in \mathfrak{h}^{k+\alpha}_\tau$, equation* (3.9) *holds.*

Proof. $(\mathbb{CH}^m, g_B)$ is a smooth complete Riemannian manifold with infinite injectivity radius. The Ricci curvature of g_B and all its covariant derivatives are bounded. So C^∞_c tensor fields are dense in $W^2_1(\mathbb{CH}^m)$ [8, 47]. By Lemma 3.12, $\mathfrak{h}^{k+\alpha}_\tau \hookrightarrow W^2_1(\mathbb{CH}^m)$ for any $k, m \in \mathbb{N}$ and $\alpha \in (0,1)$. So equation (3.9) holds for all $h \in \mathfrak{h}^{k+\alpha}_\tau$ by strong convergence in W^2_1-norm. $\square$

Thus, we extend Proposition 3.8 to the following.

Corollary 3.14. *Let $k, m \in \mathbb{N}$, $m \geq 2$, $\alpha \in (0,1)$, and fix $\tau > m/2$. For all $h \in \mathfrak{h}^{k+\alpha}_\tau \setminus \{0\}$, the curvature-normalized Ricci-DeTurck flow* (3.5) *is strictly linearly stable at $(\mathbb{CH}^m, g_B)$ in the sense of Proposition 3.8.*

Consider two Banach spaces X, Y with $Y \overset{d}{\hookrightarrow} X$, where $\overset{d}{\hookrightarrow}$ denotes a continuous and dense embedding. Given a real number $\theta \in (0,1)$, one can define the *continuous* interpolation space $(X,Y)_\theta$. The interpolation methods are well known in the literature, see for example [2, 65, 77], and we will recall their precise definitions in Section 3.7. We now state two interpolation results for the weighted little Hölder spaces $\mathfrak{h}^{k+\alpha}_\tau$. We postpone their proofs to Section 3.7. Analogous results hold for the little Hölder spaces $\mathfrak{h}^{k+\alpha}$ [79] and have been used in [41, 56, 55].

Lemma 3.15. *Let $m \in \mathbb{N}$ and fix $\tau > m/2$. Let $0 \leq k \leq \ell$ be integers, $0 < \alpha \leq \beta < 1$, and $0 < \theta < 1$, then there exists a constant $C(\theta) > 0$ depending on θ such that for all $h \in \mathfrak{h}^{\ell+\beta}_\tau$,*

$$\|h\|_{(\mathfrak{h}^{k+\alpha}_\tau,\, \mathfrak{h}^{\ell+\beta}_\tau)_\theta} \leq C(\theta)\|h\|^{1-\theta}_{\mathfrak{h}^{k+\alpha}_\tau}\|h\|^{\theta}_{\mathfrak{h}^{\ell+\beta}_\tau}. \tag{3.12}$$

Theorem 3.4. *Under the assumptions of Lemma 3.15, if $(1-\theta)(k+\alpha)+\theta(\ell+\beta) \notin \mathbb{N}$, then there is a Banach space isomorphism*

$$(\mathfrak{h}^{k+\alpha}_\tau, \mathfrak{h}^{\ell+\beta}_\tau)_\theta \cong \mathfrak{h}^{(1-\theta)(k+\alpha)+\theta(\ell+\beta)}_\tau,$$

with equivalence of the respective norms.

Consequently, $\mathfrak{h}^{\ell+\beta}_\tau \overset{d}{\hookrightarrow} (\mathfrak{h}^{k+\alpha}_\tau, \mathfrak{h}^{\ell+\beta}_\tau)_\theta \overset{d}{\hookrightarrow} \mathfrak{h}^{k+\alpha}_\tau$ for $\ell \geq k$, $\beta \geq \alpha$.

3.5 Dynamical stability of $(\mathbb{CH}^m, g_B)$

In this section, $|\cdot|$, $\|\cdot\|$, ∇, Γ^k_{ij}, and the distance function d are computed with respect to the fixed background metric g_B, and recall $\tau > m/2$ is fixed.

For fixed $0 < \sigma < \rho < 1$, consider the following sequence of densely embedded spaces

$$\begin{array}{ccccccc} \mathbb{X}_1 & \overset{d}{\hookrightarrow} & \mathbb{E}_1 & \overset{d}{\hookrightarrow} & \mathbb{X}_0 & \overset{d}{\hookrightarrow} & \mathbb{E}_0 \\ \| & & \| & & \| & & \| \\ \mathfrak{h}_\tau^{2+\rho} & & \mathfrak{h}_\tau^{2+\sigma} & & \mathfrak{h}_\tau^{0+\rho} & & \mathfrak{h}_\tau^{0+\sigma} \end{array}.$$

Let $\frac{1}{2} \leq \beta < \alpha < 1 - \frac{\rho}{2}$ be fixed, define

$$\mathbb{X}_\alpha := (\mathbb{X}_0, \mathbb{X}_1)_\alpha, \quad \mathbb{X}_\beta := (\mathbb{X}_0, \mathbb{X}_1)_\beta.$$

By the interpolation theorem (Theorem 3.4), $\mathbb{X}_\alpha \cong \mathfrak{h}_\tau^{2\alpha+\rho}$ and $\mathbb{X}_\beta \cong \mathfrak{h}_\tau^{2\beta+\rho}$ with equivalence of the respective norms. Note that $\mathbb{X}_\alpha \overset{d}{\hookrightarrow} \mathbb{X}_\beta \overset{d}{\hookrightarrow} \mathfrak{h}_\tau^{1+\rho}$.

We abbreviate the right hand side of equation (3.5) by $Q_{g_B}(g)g$. $Q_{g_B}(g) : \mathcal{S}^2_+ \to \mathcal{S}^2$ is a quasilinear elliptic operator. By a straightforward computation as in [41, Lemma 3.1], we have the following lemma.

Lemma 3.16. *If we express $Q_{g_B}(g)g$ in terms of the first and second derivatives of g in local coordinates, then*

$$\begin{aligned} (Q_{g_B}(g)g)_{ij} = {} & a(x, g_B, g_B^{-1}, g, g^{-1})_{ij}^{k\ell pq} \partial_p \partial_q g_{k\ell} \\ & + b(x, g_B, g_B^{-1}, \partial g_B, g, g^{-1}, \partial g)_{ij}^{k\ell p} \partial_p g_{k\ell} \\ & + c(x, g_B^{-1}, \partial g_B, \partial^2 g_B, g)_{ij}^{k\ell} g_{k\ell}. \end{aligned} \tag{3.13}$$

The coefficients a, b, c depend smoothly on $x \in M^n$, and they are analytic functions of their remaining arguments.

The same is true for the operator $Q_{g_0}(g)$ by replacing g_B with g_0 in equation (3.13).

Remark 3.17. *We point out two typos in [41, Lemma 3.1]: the coefficient b missed the dependence on ∂g, and the coefficient c missed the dependence on $\partial^2 g_0$.*

For a fixed $\hat{g} \in \mathbb{X}_\beta$, equation (3.13) allows us to view $Q_{g_B}(\hat{g})$ as a linear operator on $\mathbb{X}_1$: if $h \in \mathbb{X}_1$, then

$$\begin{aligned}(Q_{g_B}(\hat{g})h)_{ij} = a(x, g_B, g_B^{-1}, \hat{g}, \hat{g}^{-1})_{ij}^{k\ell pq}\partial_p\partial_q h_{k\ell} \\ + b(x, g_B, g_B^{-1}, \partial g_B, \hat{g}, \hat{g}^{-1}, \partial\hat{g})_{ij}^{k\ell p}\partial_p h_{k\ell} \\ + c(x, g_B^{-1}, \partial g_B, \partial^2 g_B, \hat{g})_{ij}^{k\ell} h_{k\ell}.\end{aligned} \tag{3.14}$$

In fact, $Q_{g_B}(g_B)$ is Δ_{g_B} plus lower order terms with bounded coefficients, and in particular, $Q_{g_B}(g_B)g_B = 0$. In the γ-basis defined in Section 3.3.2 and for fixed i, j, k, ℓ, we can represent the coefficient $a_{ij}^{k\ell pq}$ of the second order term in $Q_{g_B}(g_B)$ as a matrix a_γ. If we let λ and Λ denote the smallet and the largest eigenvalues of the matrix a_γ respectively, then $\Lambda/\lambda = 1$ on any $B_R \subset \mathbb{CH}^m$. Since the coefficient a depends analytically on $\hat{g}$, if $\hat{g}$ is sufficiently close to g_B in $\mathbb{X}_\beta$, then $Q_{g_B}(\hat{g})$ satisfies $0 < \hat{c} < \Lambda/\lambda < \hat{C}$ for some constants $\hat{c}, \hat{C}$ and for all $1 \le i, j, k, \ell \le n = 2m$. We call such $\hat{g}$ an *admissible perturbation* of g_B.

We denote by $L_{\hat{g}} := Q_{g_B}(\hat{g})$ the unbounded linear operator on $\mathbb{X}_0$ with dense domain $D(L_{\hat{g}}) = \mathbb{X}_1$. We extend $L_{\hat{g}}$ to $\hat{L}_{\hat{g}}$, which is now defined on $\mathbb{E}_0$ with dense domain $D(\hat{L}_{\hat{g}}) = \mathbb{E}_1$. If X, Y are two Banach spaces, we denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from X to Y.

For $0 < \epsilon \ll 1$ to be chosen, cf. Lemma 3.19, we define an open set in

$\mathbb{X}_\beta$ by

$$\mathbb{G}_\beta := \{g \in \mathbb{X}_\beta \text{ is an admissible perturbation} : g > \epsilon g_B\},$$

and define

$$\mathbb{G}_\alpha := \mathbb{G}_\beta \cap \mathbb{X}_\alpha.$$

Lemma 3.18.

1. *$\hat{g} \mapsto L_{\hat{g}}$ is an analytic map $\mathbb{G}_\beta \to \mathcal{L}(\mathbb{X}_1, \mathbb{X}_0)$.*

2. *$\hat{g} \mapsto \hat{L}_{\hat{g}}$ is an analytic map $\mathbb{G}_\alpha \to \mathcal{L}(\mathbb{E}_1, \mathbb{E}_0)$.*

Proof. Let $\hat{g} \in \mathbb{G}_\beta$ be given, we abbreviate equation (3.14) as

$$Q_{g_B}(\hat{g})h = a(x, \hat{g}) * \partial^2 h + b(x, \hat{g}, \partial\hat{g}) * \partial h + c(x, \hat{g}) * h,$$

where a, b, c are all analytic in $\hat{g}$, and are polynomials in $\hat{g}$, $\hat{g}^{-1}$, g_B, g_B^{-1}, and possibly in ∂g_B, $\partial^2 g_B$. $\hat{g}^{-1} \in \mathbb{G}_\beta$ implies $\hat{g}^{-1}$ is controlled by g_B^{-1}. In what follows, π denotes a polynomial that may change from line to line.

For $x, y \in A_N$ and $x \neq y$, we always have $d^N_{x,y} \leq 2$. There are three possibilities:

Case 1: $d^N_{x,y} \geq 1/2$,

Case 2: $d^N_x, d^N_y < 1/2$,

Case 3: without loss of generality, $d^N_x < 1/2 \leq d^N_y$.

If we are in Case 2, either $x, y \in A_{N-1}$ with $d_{x,y}^{N-1} \geq 1$, or $x, y \in A_{N+1}$ with $d_{x,y}^{N+1} \geq 1$, then the estimates for Case 1 will apply to Case 2 up to a different constant. If we are in Case 3, then there exists $z \in A_N$ with $d_z^N = 1/2$ such that either $x, z \in A_{N-1}, d_{x,z}^{N-1} \geq 1$ and $z, y \in A_N, d_{z,y}^N \geq 1/2$, or $x, z \in A_{N+1}, d_{x,z}^{N+1} \geq 1$ and $z, y \in A_N, d_{z,y}^N \geq 1/2$. So using the triangle inequality, the estimates for Case 1 will apply to Case 3 up to a different constant. Thus, it suffices to prove the lemma for Case 1.

We now check for Case 1. Consider $x \neq y \in A_N$ with $1/2 \leq d_{x,y}^N \leq 2$. Writing $d_{x,y}^N$ as $d_{x,y}$ for short, we estimate

$$d_{x,y}^\rho \frac{|a(x,\hat{g}) * \partial^2 h(x) - a(y,\hat{g}) * \partial^2 h(y)|}{d(x,y)^\rho} \leq d_{x,y}^\rho (F_1 + F_2),$$

where

$$F_1 = \frac{|a(x,\hat{g}) * \partial^2 h(x) - a(y,\hat{g}) * \partial^2 h(x)|}{d(x,y)^\rho},$$

$$F_2 = \frac{|a(y,\hat{g}) * \partial^2 h(x) - a(y,\hat{g}) * \partial^2 h(y)|}{d(x,y)^\rho}.$$

Since $1/2 \leq d_{x,y} \leq 2$ and $\mathbb{X}_\beta \overset{d}{\hookrightarrow} \mathfrak{h}_\tau^{1+\rho} \overset{d}{\hookrightarrow} \mathfrak{h}_\tau^{0+\rho}$, we have

$$\begin{aligned} d_{x,y}^\rho F_1 &\leq d_{x,y}^\rho \left(d_{x,y}^{-\rho} [a(\hat{g})]'_{0+\rho;A_N} d_x^{-2} |h|'_{2;A_N} \right) \\ &\lesssim e^{-2N\tau} \|\pi(\hat{g})\|_{0+\rho;\tau} \|h\|_{2+\rho;\tau} \\ &\lesssim e^{-2N\tau} \pi(\|\hat{g}\|_{\mathbb{X}_\beta}) \|h\|_{2+\rho;\tau}, \end{aligned}$$

and

$$
\begin{aligned}
d_{x,y}^{\rho} F_2 &\leq d_{x,y}^{\rho} \left(|a(\hat{g})|'_{0;A_N} d_{x,y}^{-2-\rho} [h]'_{2+\rho;A_N} \right) \\
&\lesssim e^{-2N\tau} \|\pi(\hat{g})\|_{0+\rho;\tau} \|h\|_{2+\rho;\tau} \\
&\lesssim e^{-2N\tau} \pi(\|\hat{g}\|_{\mathbb{X}_\beta}) \|h\|_{2+\rho;\tau}.
\end{aligned}
$$

So we have

$$
\sup_{N\in\mathbb{N}} e^{N\tau} [h]'_{0+\rho;A_N} \lesssim \pi(\|\hat{g}\|_{\mathbb{X}_\beta}) \|h\|_{2+\rho;\tau}.
$$

Likewise, we have

$$
\sup_{N\in\mathbb{N}} e^{N\tau} |h|'_{0;A_N} \lesssim \pi(\|\hat{g}\|_{\mathbb{X}_\beta}) \|h\|_{2+\rho;\tau}.
$$

The above estimates imply

$$
\|a(x,\hat{g}) * \partial^2 h\|_{0+\rho;\tau} \lesssim \pi(\|\hat{g}\|_{\mathbb{X}_\beta}) \|h\|_{2+\rho;\tau}.
$$

Similarly, we have

$$
\begin{aligned}
\|b(x,\hat{g},\partial\hat{g}) * \partial h\|_{0+\rho;\tau} &\lesssim \|b(x,\hat{g},\partial\hat{g})\|_{0+\rho;\tau} \|h\|_{1+\rho;\tau} \\
&\lesssim \pi(\|\hat{g}\|_{0+\rho;\tau} + \|\hat{g}\|_{1+\rho;\tau}) \|h\|_{2+\rho;\tau} \\
&\lesssim \pi(\|\hat{g}\|_{\mathbb{X}_\beta}) \|h\|_{2+\rho;\tau}, \\
\|c(x,\hat{g}) * h\|_{0+\rho;\tau} &\lesssim \pi(\|\hat{g}\|_{\mathbb{X}_\beta}) \|h\|_{2+\rho;\tau}.
\end{aligned}
$$

Putting everything together, we see

$$
\|L_{\hat{g}} h\|_{\mathbb{X}_0} = \|Q_{g_B}(\hat{g}) h\|_{\mathbb{X}_0} \lesssim \|h\|_{\mathbb{X}_1}.
$$

Thus, $\hat{g} \mapsto L_{\hat{g}}$ is an analytic map from $\mathbb{G}_\beta$ to $\mathcal{L}(\mathbb{X}_1, \mathbb{X}_0)$, which proves part (1). Part (2) is proved analogously. $\square$

Given a Banach space X, a linear operator $A : D(A) \subset X \to X$ is called *sectorial* if there are constants $\omega \in \mathbb{R}$, $\beta \in (\pi/2, \pi)$, and $M > 0$ such that

1. The resolvent set $\rho(A)$ contains
$$S_{\beta,\omega} := \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \beta\},$$
2. $\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X,X)} \leq \frac{M}{|\lambda-\omega|}$, for all $\lambda \in S_{\beta,\omega}$.

Lemma 3.19. *$\hat{L}_{g_B}$ is sectorial, and there exists $\epsilon > 0$ in the definition of $\mathbb{G}_\beta$ such that for each $\hat{g} \in \mathbb{G}_\alpha$, $\hat{L}_{\hat{g}}$ generates a strongly continuous analytic semigroup on $\mathcal{L}(\mathbb{E}_0, \mathbb{E}_0)$.*

Proof. Since $\hat{L}_{g_B}$ is Δ_{g_B} plus lower order terms, it is a strongly elliptic operator. The spectrum of Δ_{g_B} on $\mathbb{E}_1$ is $(\infty, 0]$, so that $\mathrm{Spec}(L_{g_B}) \subset (-\infty, \lambda_0)$ for some $\lambda_0 \in \mathbb{R}$, since the lower order terms may affect the spectrum by a bounded amount. Then by an argument similar to the proof of [41, Lemma 3.4], $\hat{L}_{g_B}$ is sectorial by the Schauder estimates for $\hat{L}_{g_B}$ with respect to the $\mathfrak{h}_\tau^{k+\rho}$-norms. To see such estimates hold, one first uses the standard Schauder estimates for $\hat{L}_{g_B}$ on A_N that
$$\sum_{\ell=0}^{2} |h|'_{\ell;A_N} + [h]'_{2+\rho;A_N} \leq C\left(|\hat{L}_{g_B} h|'_{0;A_N} + [\hat{L}_{g_B} h]'_{\rho;A_N}\right),$$
where C is a constant depending on the complex dimension m, the Hölder exponent ρ, and the ratio Λ/λ, but not on N [38]. Multiplying both sides of

this inequality by $e^{N\tau}$ and taking the supremum over $N \in \mathbb{N}$, then

$$\|h\|_{2+\rho;\tau} \lesssim \|\hat{L}_{g_B} h\|_{0+\rho;\tau}.$$

So $\hat{L}_{g_B}$ is sectorial, and since $\hat{L}_{g_B}$ is densely defined by construction, it generates a strongly continuous analytic semigroup by a standard characterization [64, pp. 34]. By part (2) of Lemma 3.18, we can choose $\epsilon > 0$ in the definition of $\mathbb{G}_\beta$ so small that for $\hat{g} \in \mathbb{G}_\alpha$, we have

$$\|\hat{L}_{\hat{g}} - \hat{L}_{g_B}\|_{\mathcal{L}(\mathbb{E}_1, \mathbb{E}_0)} < \frac{1}{M+1},$$

for the constant $M > 0$ in the definition of sectorial operator corresponding to $\hat{L}_{g_B}$. So the perturbation $\hat{L}_{\hat{g}}$ is sectorial by [64, Proposition 2.4.2], and hence generates a strongly continuous analytic semigroup on $\mathcal{L}(\mathbb{E}_0, \mathbb{E}_0)$. $\square$

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. We verify conditions (B1)–(B7) in Theorem 3.3.

(B1): By construction, $\mathbb{X}_1 \overset{d}{\hookrightarrow} \mathbb{X}_0$ and $\mathbb{E}_1 \overset{d}{\hookrightarrow} \mathbb{E}_0$ are continuous dense inclusions of Banach spaces. For fixed $\frac{1}{2} \leq \beta < \alpha < 1 - \frac{\rho}{2}$, $\mathbb{X}_\alpha, \mathbb{X}_\beta$ are continuous interpolation spaces corresponding to the inclusion $\mathbb{X}_1 \overset{d}{\hookrightarrow} \mathbb{X}_0$.

(B2): Equation (3.5) is an autonomous quasilinear parabolic equation. There exists a positive integer K such that for all $\hat{g}$ in the open set $\mathbb{G}_\beta \subset \mathbb{X}_\beta$, the domain $D(L_{\hat{g}})$ of $L_{\hat{g}}$ is equal to $\mathbb{X}_1$, and the map $\hat{g} \to L_{\hat{g}}|_{\mathbb{X}_1}$ belongs to $C^K(\mathbb{G}_\beta, \mathcal{L}(\mathbb{X}_1, \mathbb{X}_0))$ by part (1) of Lemma 3.18. In fact, we can let $K = \infty$.

(B4): By Lemma 3.19, for each $\hat{g} \in \mathbb{G}_\alpha$, $\hat{L}_{\hat{g}}|_{\mathbb{E}_1} \in \mathcal{L}(\mathbb{E}_1, \mathbb{E}_0)$ generates a strongly continuous analytic semigroup on $\mathcal{L}(\mathbb{E}_0, \mathbb{E}_0)$.

(B3), (B5): By construction, for each $\hat{g} \in \mathbb{G}_\beta$, $\hat{L}_{\hat{g}}$ is an extension of $L_{\hat{g}}$ to a domain $D(\hat{L}_{\hat{g}})$ that is equal to $\mathbb{E}_0$. For each $\hat{g} \in \mathbb{G}_\alpha, L_{\hat{g}}$ is the part of $\hat{L}_{\hat{g}}$ in $\mathbb{X}_0$.

(B6): Recall that $0 < \sigma < \rho < 1$ were fixed. For each $\hat{g} \in \mathbb{G}_\alpha$, there exists $\theta = \frac{\rho - \sigma}{2} \in (0,1)$ such that, by Theorem 3.4,

$$(\mathbb{E}_0, D(\hat{L}_{\hat{g}}))_\theta = (\mathfrak{h}_\tau^{0+\sigma}, \mathfrak{h}_\tau^{2+\sigma})_\theta \cong \mathfrak{h}_\tau^{0+\rho} = X_0$$

with equivalence of the respective norms.

Define the set $(\mathbb{E}_0, D(\hat{L}_{\hat{g}}))_{1+\theta} := \{g \in D(\hat{L}_{\hat{g}}) : \hat{L}_{\hat{g}}(g) \in (\mathbb{E}_0, D(\hat{L}_{\hat{g}}))_\theta\}$, and endow it with the graph norm of $\hat{L}_{\hat{g}}$ with respect to the space $(\mathbb{E}_0, D(\hat{L}_{\hat{g}}))_\theta$, which we just showed to be equivalent to the space $\mathbb{X}_0$. This graph norm is $\|\cdot\|_{\mathbb{X}_0} + \|\hat{L}_{\hat{g}}(\cdot)\|_{\mathbb{X}_0}$.

By definition, $\mathbb{X}_1 \overset{d}{\hookrightarrow} (\mathbb{E}_0, D(\hat{L}_{\hat{g}}))_{1+\theta}$. We claim the respective norms are equivalent. Indeed, $\|\cdot\|_{\mathbb{X}_0} + \|\hat{L}_{\hat{g}}(\cdot)\|_{\mathbb{X}_0} \lesssim \|\cdot\|_{\mathbb{X}_1}$ since $\mathbb{X}_1 \overset{d}{\hookrightarrow} \mathbb{X}_0$. The opposite inequality, $\|\cdot\|_{\mathbb{X}_1} \lesssim \|\cdot\|_{\mathbb{X}_0} + \|\hat{L}_{\hat{g}}(\cdot)\|_{\mathbb{X}_0}$, follows from Schauder estimates for $\hat{L}_{\hat{g}}$ with respect to the weighted Hölder norms $\|\cdot\|_{k+\rho;\tau}$, similar to those used in the proof of Lemma 3.19. So the claim is true. Therefore, there is a Banach space isomorphism $(\mathbb{E}_0, D(\hat{L}_{\hat{g}}))_{1+\theta} \cong \mathbb{X}_1$ with equivalence of the respective norms.

(B7): This just follows from Lemma 3.15.

To finish the proof, for fixed $\rho \in (0,1)$ and $\alpha \in (\frac{1}{2}, 1 - \frac{\rho}{2})$, there exists $\eta \in (\rho, 1)$ such that, by Theorem 3.4,

$$\mathbb{X}_\alpha \cong \mathfrak{h}_\tau^{2\alpha+\rho} =: h_\tau^{1+\eta}$$

with equivalence of the respective norms. By Corollary 3.14, $\mathrm{Spec}(A) \subset (-\infty, -(m-1)c/2]$ for $c > 0$ in complex dimension $m \geq 2$. Therefore, we can apply the Stability Theorem (Theorem 3.3) to conclude Theorem 3.1. □

3.6 Dynamical stability of (M^n, g_0)

For fixed $0 < \sigma < \rho < 1$, consider the following densely and continuously embedded spaces:

$$\begin{array}{ccccccc} \mathbb{X}_1 & \overset{d}{\hookrightarrow} & \mathbb{E}_1 & \overset{d}{\hookrightarrow} & \mathbb{X}_0 & \overset{d}{\hookrightarrow} & \mathbb{E}_0 \\ \| & & \| & & \| & & \| \\ \mathfrak{h}^{2+\rho} & & \mathfrak{h}^{2+\sigma} & & \mathfrak{h}^{0+\rho} & & \mathfrak{h}^{0+\sigma} \end{array}.$$

For fixed $1/2 \leq \beta < \alpha < 1$, define

$$\mathbb{X}_\beta := (\mathbb{X}_0, \mathbb{X}_1)_\beta, \quad \mathbb{X}_\alpha := (\mathbb{X}_0, \mathbb{X}_1)_\alpha.$$

For fixed $0 < \epsilon \ll 1$, define

$$\mathbb{G}_\beta := \{g \in \mathbb{X}_\beta : g > \epsilon g_0\}, \quad \mathbb{G}_\alpha := \mathbb{G}_\beta \cap \mathbb{X}_\alpha,$$

where "$g > \epsilon g_0$" means $g(X,X) > \epsilon g_0(X,X)$ for any tangent vector X.

We are now ready for the proof of Theorem 3.2.

Proof of Theorem 3.2. Proposition 3.8 applied to $h \in \mathbb{X}_1$ (or $h \in \mathbb{E}_1$) implies the spectrum of the operator A, denoted by $\mathrm{Spec}(A)$, satisfies $\mathrm{Spec}(A) \subset$

$(-\infty, -(m-1)c/2]$ for $c > 0$ and $m \geq 2$. We then apply the Stability Theorem (Theorem 3.3) to conclude Theorem 3.2. We omit the details since they are similar to the proof in [41]. $\square$

3.7 Interpolation properties

Let X, Y be two Banach spaces with $Y \overset{d}{\hookrightarrow} X$. We review the K-method in interpolation theory, cf. [2, 65, 79].

For every $h \in X + Y$ and $t > 0$, set

$$K(t,h) = K(t,h,X,Y) := \inf_{\substack{h=a+b,\\ a\in X,\, b\in Y}} \left(\|a\|_X + t\|b\|_Y\right).$$

Let $\theta \in (0,1)$, $p \in [1,\infty]$. We define the *real* interpolation space between X and Y by

$$(X,Y)_{\theta,p} := \{h \in X + Y : t \mapsto t^{-\theta}K(t,h) \in L^p_*(0,\infty)\},$$

where $L^p_*(0,\infty)$ is the L^p space with respect to the measure dt/t. We abbreviate $L^p_*(0,\infty)$ as L^p_*. In particular, $\|\cdot\|_{L^\infty_*} = \|\cdot\|_{L^\infty}$. The norm of $h \in (X,Y)_{\theta,p}$ is given by

$$\|h\|_{\theta,p} = \|h\|_{(X,Y)_{\theta,p}} := \|t^{-\theta}K(t,h)\|_{L^p_*}.$$

We define the *continuous* interpolation space between X and Y by

$$(X,Y)_\theta := \{h \in X + Y : \lim_{t\to 0+} t^{-\theta}K(t,h) = 0\}.$$

Since for every $h \in X + Y$, $t \mapsto K(t,h)$ is concave and hence continuous in $(0,\infty)$, $(X,Y)_\theta$ is a closed subspace of $(X,Y)_{\theta,\infty}$, and it is endowed with the $(X,Y)_{\theta,\infty}$-norm, i.e., $\|\cdot\|_\theta = \|\cdot\|_{\theta,\infty}$.

Remark 3.20. *Since* $Y \overset{d}{\hookrightarrow} X$*, we can equivalently define* $(X,Y)_\theta$ *to be the closure of* Y *in the* $(X,Y)_{\theta,\infty}$*-norm [77].*

We now prove Lemma 3.15.

Proof of Lemma 3.15. We use the general fact [65, Corollary 1.7] that given two Banach spaces X, Y with $Y \overset{d}{\hookrightarrow} X$, if $\theta \in (0,1)$ and $p \in [1,\infty]$, then there exists a constant $C(\theta,p) > 0$ such that for all $y \in Y$,

$$\|y\|_{\theta,p} \leq C(\theta,p)\|y\|_X^\theta \|y\|_Y^{1-\theta}.$$

Taking $Y := \mathfrak{h}^{\ell+\beta} \overset{d}{\hookrightarrow} \mathfrak{h}^{k+\alpha} =: X$, and noting when $p = \infty$, $\|\cdot\|_\theta = \|\cdot\|_{\theta,\infty}$, we obtain (3.12). $\square$

In the rest of this section, we prove Theorem 3.4.

Lemma 3.21. *Let* $m \in \mathbb{N}$ *and fix* $\tau > m/2$*, then for each* $\theta \in (0,1)$*, we have the Banach space isomorphism*

$$(\mathfrak{h}^0_\tau, \mathfrak{h}^1_\tau)_\theta \cong \mathfrak{h}^\theta_\tau$$

with equivalence of the respective norms.

Proof. We let $X := \mathfrak{h}^0_\tau$, $Y := \mathfrak{h}^1_\tau$, and $Z := \mathfrak{h}^\theta_\tau$. Note that $Y \overset{d}{\hookrightarrow} Z \overset{d}{\hookrightarrow} X$.

We first show $(X,Y)_\theta \subset Z$, i.e., $(\mathfrak{h}^0_\tau, \mathfrak{h}^1_\tau)_\theta \subset \mathfrak{h}^\theta_\tau$.

Let $h \in (X,Y)_\theta \subset X$. If $b \in Y \subset X$, then $a := h - b \in X$. So we can always decompose $h = a + b$ with $a \in X$ and $b \in Y$, and

$$\|h\|_X \leq \|a\|_X + \|b\|_X \lesssim \|a\|_X + \|b\|_Y,$$

since $Y \overset{d}{\hookrightarrow} X$. Taking the infimum over all such decompositions, then

$$\|h\|_X \leq K(1,h) \leq \|h\|_\theta,$$

since for $h \in (X,Y)_\theta$ we have the useful fact that for all $t > 0$, $K(t,h) \leq t^\theta \|h\|_\theta$.

If $x \neq y \in A_N$ for some N, then we can find a curve γ in A_N connecting x and y such that $d_z \geq d_{x,y}$ for all $z \in \gamma$ and $\text{Length}(\gamma) \lesssim d(x,y)$. We have, in the single geodesic coordinate chart covering $\mathbb{CH}^m$,

$$\begin{aligned} |a(x) - a(y)| &\leq |a(x)| + |a(y)| \lesssim e^{-N\tau}\|a\|_X, \\ |b(x) - b(y)| &\leq \int_\gamma |\langle \text{grad } b, \dot{\gamma}\rangle| \,. \end{aligned}$$

Given a decomposition $h = a + b$ with $a \in X$, $b \in Y$, we see

$$\begin{aligned} d_{x,y}^\theta |h(x) - h(y)| &\leq d_{x,y}^\theta |a(x) - a(y)| + d_{x,y}^\theta |b(x) - b(y)| \\ &\lesssim d_{x,y}^\theta e^{-N\tau}\|a\|_X + \int_\gamma \frac{d_{x,y}^\theta}{d_z^\theta} d_z^\theta \,|\text{grad } b(z)| \\ &\lesssim e^{-N\tau}\|a\|_X + e^{-N\tau}\|b\|_Y d(x,y) \\ &\lesssim e^{-N\tau} K(d(x,y),h) \\ &\lesssim e^{-N\tau} d(x,y)^\theta \|h\|_\theta. \end{aligned}$$

So $[h]'_{0+\theta;A_N} \leq e^{-N\tau}\|h\|_\theta$, and it follows that

$$\begin{aligned} \|h\|_Z &= \|h\|_X + \sup_{N\in\mathbb{N}} e^{N\tau}[h]'_{0+\theta;A_N} \\ &\lesssim \|h\|_\theta + \|h\|_\theta. \end{aligned}$$

Therefore, $h \in Z$ with $\|h\|_Z \lesssim \|h\|_\theta$.

We now show $(X,Y)_\theta \supset Z$, i.e., $(\mathfrak{h}^0_\tau, \mathfrak{h}^1_\tau)_\theta \supset \mathfrak{h}^\theta_\tau$.

Let $h \in Z \subset X$. We will suitably decompose $h = a_t + b_t$ for $t \in (0,\infty)$ with $a_t \in X$, $b_t \in Y$. When $t \in [1,\infty)$, let $a_t = h$, $b_t = 0$, then $K(t,h) \le \|h\|_X$, and hence

$$t^{-\theta} K(t,h) \le t^{-\theta}\|h\|_X \le \|h\|_X \lesssim \|h\|_Z$$

because $Z \overset{d}{\hookrightarrow} X$.

We now consider the case $t \in (0,1]$.

For each fixed $x \in \mathbb{CH}^m$, let $\zeta(y) = \zeta\left(d(x,y)\right)$ be a smooth positive bump function compactly supported in $\{y \in \mathbb{CH}^m : d(x,y) \le 1\}$ with unit mass:

$$1 = \int_{\mathbb{CH}^m} \zeta\left(d(x,y)\right) d\mu_{g_B}(y).$$

Let B_t denote a geodesic ball of radius t in $\mathbb{CH}^m$ and $V(B_t)$ its volume. Recall that g_B has constant scalar curvature $R < 0$, and that the asymptotic expansion of $V(B_t)$ with respect to t is

$$\frac{V(B_t)}{\omega_{2m} t^{2m}} = 1 - \frac{R}{6(2m+2)} t^2 + O(t^3),$$

where ω_{2m} is the volume of the unit ball in Euclidean $\mathbb{R}^{2m}$. Then under the homothetic scaling $d(x,y) \mapsto d(x,y)/t$, there exist constants $\bar{c}, \bar{C} > 0$ such that for $t \in (0,1]$,

$$\bar{c} \le C_t := \frac{1}{t^n} \int_{\{d(x,y)\le t\}} \zeta\left(\frac{d(x,y)}{t}\right) d\mu_{g_B}(y) \le \bar{C}.$$

Define

$$b_t(x) := \frac{1}{C_t}\left[\frac{1}{t^n}\int_{\{d(x,y)\le t\}} h(y)\zeta\left(\frac{d(x,y)}{t}\right)d\mu_{g_B}(y)\right],$$
$$a_t(x) := h(x) - b_t(x).$$

For fixed $x \in \mathbb{CH}^m$ and $y \in \{d(x,y) \le t \le 1\}$, we can find some A_N such that $B_1(x) \subset A_N$ with $d_{x,y} \ge 1/2$, so then $|h(x)-h(y)|/d(x,y)^\theta \le e^{-N\tau}d_{x,y}^{-\theta}\|h\|_Z \lesssim e^{-N\tau}\|h\|_Z$.

We now estimate

$$\begin{aligned}
|a_t(x)| = |h(x) - b_t(x)| &= \left|\frac{C_t}{C_t}h(x) - b_t(x)\right| \\
&\le \frac{1}{C_t}\frac{1}{t^n}\int_{\{d(x,y)\le t\}} |h(x)-h(y)|\left|\zeta\left(\frac{d(x,y)}{t}\right)\right| d\mu_{g_B}(y) \\
&= \frac{1}{C_t}\frac{1}{t^n}\int_{\{0<d(x,y)\le t\}} \frac{|h(x)-h(y)|}{d(x,y)^\theta}d(x,y)^\theta\left|\zeta\left(\frac{d(x,y)}{t}\right)\right| d\mu_{g_B}(y) \\
&\lesssim \frac{1}{t^n}\int_{\{d(x,y)\le t\}} e^{-N\tau}\|h\|_Z d(x,y)^\theta\left|\zeta\left(\frac{d(x,y)}{t}\right)\right| d\mu_{g_B}(y) \\
&= e^{-N\tau}\|h\|_Z\int_{\{d(x,y)\le t\}} \frac{1}{t^n}d(x,y)^\theta\left|\zeta\left(\frac{d(x,y)}{t}\right)\right| d\mu_{g_B}(y).
\end{aligned}$$

Recall the definition $\|a_t\|_X = \sup\limits_{N\in\mathbb{N}} e^{N\tau}|a_t(x)|'_{0;A_N}$ and using $d(x,y)/t \le 1$, then

$$\|a_t\|_X \lesssim \|h\|_Z \int_{\{d(x,y)\le t\}} \frac{t^\theta}{t^n}\left|\zeta\left(\frac{d(x,y)}{t}\right)\right| d\mu_{g_B}(y). \tag{3.15}$$

Analogously, we have

$$\sup_{N\in\mathbb{N}} e^{N\tau}|b_t(x)|'_{1;A_R} \lesssim \|h\|_Z \int_{\{d(x,y)\le t\}} \frac{t^{\theta-1}}{t^n}\left|\partial\zeta\left(\frac{d(x,y)}{t}\right)\right| d\mu_{g_B}(y). \tag{3.16}$$

Since $Z \overset{d}{\hookrightarrow} X$, $\theta \in (0,1)$ is fixed, and $t \in (0,1]$, we have

$$\begin{aligned}
t^{1-\theta}\|b_t\|_X &= t^{1-\theta}\|h - a_t\|_X \\
&\leq t^{1-\theta}\left(\|h\|_X + \|a_t\|_X\right) \\
&\lesssim \|h\|_Z \left(1 + t\int_{\{d(x,y)\leq t\}} \left|\zeta\left(\frac{d(x,y)}{t}\right)\right| d\mu_{g_B}(y)\right). \qquad (3.17)
\end{aligned}$$

Putting together estimates (3.15), (3.16), and (3.17), for $t \in (0,1]$,

$$\begin{aligned}
t^{-\theta}K(t,h) &\leq t^{-\theta}\left(\|a_t\|_X + t\|b_t\|_Y\right) \\
&= t^{-\theta}\left(\|a_t\|_X + t\|b_t\|_X + t\sup_{N\in\mathbb{N}} e^{N\tau}|b_t(x)|'_{1;A_N}\right) \\
&\lesssim \|h\|_Z \\
&\quad + \|h\|_Z \int_{\{d(x,y)\leq t\}} \left(\left|\zeta\left(\frac{d(x,y)}{t}\right)\right| + \left|\partial\zeta\left(\frac{d(x,y)}{t}\right)\right|\right) d\mu_{g_B}(y) \\
&\lesssim \|h\|_Z.
\end{aligned}$$

Thus, we obtain for $t \in (0,1]$,

$$t^{-\theta}K(t,h) \lesssim \|h\|_Z.$$

To conclude $h \in (X,Y)_\theta$, it remains to show that $\lim_{t\to 0+} t^{-\theta}K(t,h) = 0$.

Fix $x \in \mathbb{CH}^m$, let $t \in (0,1]$ and $y \in \{d(x,y) \leq t\}$, then $x, y \in A_N$ for some $N \in \mathbb{N}$ with $d_{x,y} \geq 1/2$. Noting $d(x,y)/t \leq 1$, we estimate

$$
\begin{aligned}
t^{-\theta}|a_t(x)| &\leq t^{-\theta} \frac{1}{C_t} \frac{1}{t^n} \int_{\{d(x,y) \leq t\}} |h(x) - h(y)| \left| \zeta \left(\frac{d(x,y)}{t} \right) \right| d\mu_{g_B}(y) \\
&= \frac{1}{C_t} \frac{1}{t^n} \int_{\{0 < d(x,y) \leq t\}} \frac{d_{x,y}^\theta}{d_{x,y}^\theta} \frac{|h(x) - h(y)|}{d(x,y)^\theta} \left(\frac{d(x,y)}{t} \right)^\theta \left| \zeta \left(\frac{d(x,y)}{t} \right) \right| d\mu_{g_B}(y) \\
&\lesssim \frac{1}{C_t} \frac{1}{t^n} \int_{\{0 < d(x,y) \leq t\}} d_{x,y}^\theta \frac{|h(x) - h(y)|}{d(x,y)^\theta} \left| \zeta \left(\frac{d(x,y)}{t} \right) \right| d\mu_{g_B}(y) \\
&\leq \sup_{\substack{x \neq y \in A_N, \\ d(x,y) \leq t}} d_{x,y}^\theta \frac{|h(x) - h(y)|}{d(x,y)^\theta} \left(\frac{1}{C_t} \int_{\{d(x,y) \leq t\}} \frac{1}{t^n} \left| \zeta \left(\frac{d(x,y)}{t} \right) \right| d\mu_{g_B}(y) \right) \\
&= \sup_{\substack{x \neq y \in A_N, \\ d(x,y) \leq t}} d_{x,y}^\theta \frac{|h(x) - h(y)|}{d(x,y)^\theta}.
\end{aligned}
$$

As $t \to 0+$, by property (3.11) that for $h \in Z := \mathfrak{h}_\tau^\theta$,

$$
\lim_{t \to 0+} \sup_{N \in \mathbb{N}} e^{N\tau} \sup_{\substack{x \neq y \in A_N, \\ d(x,y) \leq t}} d_{x,y}^\theta \frac{|h(x) - h(y)|}{d(x,y)^\theta} = 0,
$$

hence

$$
\lim_{t \to 0+} t^{-\theta} \|a_t\|_X = 0.
$$

Similarly, $\lim\limits_{t \to 0+} t^{1-\theta} \|b_t\|_Y = 0$. Thus, $\lim\limits_{t \to 0+} t^{-\theta} K(t,h) = 0$.

Therefore, $h \in (X,Y)_\theta$ with $\|h\|_\theta \lesssim \|h\|_Z$. $\square$

Let X be a Banach space. Consider a linear operator $A : D(A) \to X$ such that

$$
\begin{aligned}
&(0,\infty) \subset \rho(A), \text{ and there exists a constant } C > 0 \text{ such that} \\
&\qquad \|\lambda R(\lambda, A)\|_{\mathcal{L}(X,X)} \leq C, \text{ for all } \lambda > 0,
\end{aligned} \tag{3.18}
$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda, A) = (\lambda I - A)^{-1}$ is the resolvent operator of A. Note this is condition (3.1) in [65].

Lemma 3.22. *Let $m \in \mathbb{N}$ and fix $\tau > m/2$. For $1 \leq i \leq n = 2m$, consider $A_i := \partial_i$ in geodesic normal coordinates with respect to g_B. Then for each i, $A_i := D(A_i) : \mathfrak{h}^1_\tau \to \mathfrak{h}^0_\tau$ satisfies condition* (3.18).

Proof. Let $h \in \mathfrak{h}^0_\tau$. Write $x = (x^1, \ldots, x^n) \in \mathbb{CH}^m$ in the single geodesic normal coordinates, so $0 \leq x^i < \infty$, $1 \leq i \leq n$. Since $(\mathbb{CH}^m, g_B)$ is a complete manifold, there exists for each $1 \leq i \leq n$ a unique geodesic $\gamma_i : \mathbb{R} \to \mathbb{CH}^m$ with $\gamma_i(x^i) = x$ and $\dot{\gamma}(x^i) = \partial/\partial x^i|_x$. Then for each $\lambda > 0$,

$$(R(\lambda, A_i)h)(x) := \int_{x^i}^{\infty} e^{-\lambda(t - x^i)} h(\gamma_i(t)) dt$$

is the resolvent operator of A_i, $1 \leq i \leq n$. In particular, $(0, \infty) \subset \rho(A)$.

If $i = 1$, then $M - 1 \leq x^1 \leq M$ since $x \in B_M \setminus B_{M-1}$ for some $M \in \mathbb{N}$, which corresponds to $x \in A_N$ for some $N = N(M) \in \mathbb{N}$. We then estimate

$$\begin{aligned}
e^{N\tau} |(\lambda R(\lambda, A_1)h)(x)| &\leq e^{N\tau} \sum_{K=N}^{\infty} \int_{K-1}^{K+3} \lambda e^{-\lambda(t - x^1)} |h(\gamma_i(t))| dt \\
&\leq \sum_{K=N}^{\infty} \int_{K-1}^{K+3} \lambda e^{-\lambda(t - x^1)} e^{N\tau} e^{-K\tau} \|h\|_{0;\tau} dt \\
&\leq \|h\|_{0;\tau} \sum_{K=N}^{\infty} \int_{K-1}^{K+3} \lambda e^{-\lambda(t - x^1)} dt \\
&\leq 4\|h\|_{0;\tau} \int_{M-1}^{\infty} \lambda e^{-\lambda(t - x^1)} dt \\
&\leq 4\|h\|_{0;\tau} e^{-\lambda(M - 1 - x^1)} \\
&\leq C_1 \|h\|_{0;\tau}.
\end{aligned}$$

Similarly, for $2 \le i \le n$,

$$e^{N\tau}|(\lambda R(\lambda, A_i)h)(x)| \le C_i \|h\|_{0;\tau}.$$

Taking the supremum over $N \in \mathbb{N}$, then for each $\lambda > 0$,

$$\|\lambda R(\lambda, A_i)h\|_{0;\tau} \le \max_{1\le i\le n} C_i \|h\|_{0;\tau}.$$

Therefore, A_i satisfies condition (3.18) for $1 \le i \le n$. $\square$

Lemma 3.22 ensures the interpolation theory in [65] applies, so from the basic step Lemma 3.21 we can conclude the following.

Lemma 3.23. *Let $m \in \mathbb{N}$ and fix $\tau > m/2$, then for $\theta \in (0,1)$ and $\ell \in \mathbb{N}$ such that $\ell\theta \notin \mathbb{N}$, we have the Banach space isomorphism*

$$(\mathfrak{h}^0_\tau, \mathfrak{h}^\ell_\tau)_\theta \cong \mathfrak{h}^{\ell\theta}_\tau$$

with equivalence of the respective norms.

We can now prove Theorem 3.4.

Proof of Theorem 3.4. Given $\mathfrak{h}^{k+\alpha}_\tau$, $\mathfrak{h}^{\ell+\beta}_\tau$, there exist $\theta_0, \theta_1 \in (0,1)$ and $p \in \mathbb{N}$ such that by Lemma 3.23,

$$(\mathfrak{h}^0_\tau, \mathfrak{h}^p_\tau)_{\theta_0} \cong \mathfrak{h}^{p\theta_0}_\tau = \mathfrak{h}^{k+\alpha}_\tau,$$
$$(\mathfrak{h}^0_\tau, \mathfrak{h}^p_\tau)_{\theta_1} \cong \mathfrak{h}^{p\theta_1}_\tau = \mathfrak{h}^{\ell+\beta}_\tau,$$

with equivalence of the respective norms.

Suppose $(1-\theta)(k+\alpha)+\theta(\ell+\beta) \notin \mathbb{N}$, then

$$\begin{aligned}
(\mathfrak{h}_\tau^{k+\alpha}, \mathfrak{h}_\tau^{\ell+\beta})_\theta &\cong \big((\mathfrak{h}_\tau^0, \mathfrak{h}_\tau^p)_{\theta_0}, (\mathfrak{h}_\tau^0, \mathfrak{h}_\tau^p)_{\theta_1}\big)_\theta \\
&\cong (\mathfrak{h}_\tau^0, \mathfrak{h}_\tau^p)_{(1-\theta)\theta_0+\theta\theta_1} \quad \text{(by the Reiteration Theorem [79, 65])} \\
&\cong \mathfrak{h}_\tau^{(1-\theta)p\theta_0+\theta p\theta_1} \quad \text{(by Lemma 3.23)} \\
&= \mathfrak{h}_\tau^{(1-\theta)(k+\alpha)+\theta(\ell+\beta)}.
\end{aligned}$$

with equivalence of the respective norms. □

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